String without strings

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Abstract

Scale invariance provides a principled reason for the physical importance of Hilbert space, the Virasoro algebra, the string mode expansion, canonical commutators and Schroedinger evolution of states, independent of the assumptions of string theory and quantum theory. The usual properties of dimensionful fields imply an infinite, projective tower of conformal weights associated with the tangent space to scale-invariant spacetimes. Convergence and measurability on this tangent tower are guaranteed using a scale-invariant norm, restricted to conformally self-dual vectors. Maps on the resulting Hilbert space are correspondingly restricted to semi-definite conformal weight. We find the maximally- and minimally-commuting, complete Lie algebras of definite-weight operators. The projective symmetry of the tower gives these algebras central charges, giving the canonical commutator and quantum Virasoro algebras, respectively. Using a continuous, m-parameter representation for rank-m tower tensors, we show that the parallel transport equation for the momentum vector of a particle is the Schroedinger equation, while the associated definite-weight operators obey canonical commutation relations. Generalizing to the set of integral curves of general timelike, self-dual vector-valued weight maps gives a lifting such that the action of the curves parallel transports arbitrary tower vectors. We prove that the full set of Schroedinger-lifted integral curves of a general self-dual map gives an immersion of its 2-dim parameter space into spacetime, inducing a Lorentzian metric on the parameter space. This immersion is shown to satisfy the full variational equations of open string.

1   Introduction

It is certainly fair to say that the string mode operators and the associated Virasoro algebra, acting on Hilbert space, form the heart and soul of string theory. States can be classified by mass and spin because we can locally construct generators of the Poincaré group from the mode operators; this is also the reason the effective spin-2 field equations must be generally covariant. The action of the Virasoro algebra determines the space of physical states, and therefore is responsible for the spin-2 (gravitational) mode present in the closed string ground state. Having a generally covariant, massless, spin-2 mode is a necessary (though not sufficient) condition for string theory to approximate general relativity. Upon quantization, it is the central charge of the Virasoro algebra that fixes the dimension of the target space at 26 in the bosonic theory or 10 in the supersymmetric case. And it is the existence of supersymmetric and heterotic representations of the basic symmetries that frees string theory from tachyons, guarantees fermions and provides a large internal nonabelian gauge symmetry.

It is therefore remarkable to discover that exactly this configuration - the mode algebra and the Virasoro algebra with central charges, acting on Hilbert space - *occurs classically in the tangent space to every scale-invariant geometry.* Hilbert space occurs as the tower of conformal weights, while the mode and Virasoro algebras occur as the unique complete Lie algebras of maximally or minimally commuting, definite-weight operators.

The missing element in recognizing this fact earlier is the failure to see the extended character of the tangent space of scale invariant geometries. The different scaling behavior of fields of different dimension leads to the association of a *conformal weight* with each field. These conformal weights are associated with an equivalence relation on the tangent space at each point of spacetime. The usual Minkowski tangent space modulo this equivalence relation gives the *tangent tower*, comprised of a copy of projective Minkowski space for each possible weight. The tangent tower has a natural scale-invariant inner product under which the conformally self-dual *weight vectors* form a Hilbert space. A study of linear operators on the tangent tower reveals the presence of the mode operator and Virasoro algebras, while the existence of nontrivial projective representations gives rise to central charges.

While the spontaneous breaking of conformal invariance in field theory is well known, the scaling symmetry described here remains *unbroken.* To see clearly the difference between the two symmetries, consider a typical example of spontaneous symmetry breaking. It is common for an initially scale-invariant Lagrangian to develop one-loop corrections of the general form

$$\lambda^2 \ln(\frac{\Lambda}{\mu}) \tag{1}$$

where μ is fixed by a particle mass. Often, the necessary inclusion of the dimensionful parameter Λ breaks the original scaling symmetry. However, the dimensionality of Λ is

determined by a much deeper, unbreakable symmetry - the matching of the units of Λ to the units of μ. Reliance on the inviolable balance of units is how we knew the dimensions of Λ to begin with. It is this balance of units which we refer to here as scale-invariance, a symmetry without which no physical equation makes sense. Thus, in the "conformal symmetry breaking" logarithmic expression above, the transformation $\Lambda \to e^{\varphi}\Lambda, \mu \to e^{\varphi}\mu$ is a *distinct* scale symmetry which remains unbroken. It is remarkable that the simple fact of using units to describe physical quantities implies the rich formalism of the weight tower.

The layout of our investigation of these ideas is as follows. In Sec.(2), we present a very brief review of some core elements of string theory. Next, in Sec.(3), we discuss the properties of scale-invariant geometries, giving the simple example of a Weyl geometry for a typical context, but focussing principally on the tangent space structure common to all scaling geometries. After defining the tangent tower we study its linear transformations in order to define tensors on the tower. Restricting to definite-conformal-weight linear transformations as dictated by the requirements of measurement, we seek a Lie algebra of operators. Two are found: the maximally commuting and maximally non-commuting algebras containing one operator of each weight. The commuting algebra is trivial to derive, but a detailed proof is required to show that the maximally non-commuting, complete Lie algebra of definite-weight operators is the Virasoro algebra. The details of the proof are given in Sec.(4).

Once these algebraic structures are specified, we look at a continuous representation for tower tensors, using n continuous, bounded variables for rank-n tower tensors. We show that the zero-weight and zero-anti-weight operators together span the space of harmonic functions and find continuous representations for the Virasoro and mode algebras. Then, in Sec.(6), we look first at the effects of different possible equivalence relations in defining the tower structure, then at the effects of the projective group representations permitted by the tower symmetry, including central extensions for both the mode and Virasoro algebras.

As preparation for our study of the conformal dynamics of stringlike variables, the conformal dynamics of a point particle is sketched in Sec.(7). We show how parallel transport on the tangent tower gives the Schrodinger equation for the evolution of tower vectors. We also show how the self-duality required for convergence of tower vectors combines with the projective equivalence that defines the tower to give canonical commutation relations for conformally conjugate variables.

The results of Secs.(5),(6) and (7) are used in Sec.(8) to show an important relationship of vector-valued weight maps ($(1,2)$ tensor fields) to strings: the integral transport curves of a timelike, self-dual $(1,2)$ tensor field provide an immersion of $(0,\pi) \times (0,\pi)$ into spacetime, causal in the pull-back of the spacetime metric, such that the immersed 2-surface extremizes the Polykov string action with respect to variations of the string

coordinates, the string endpoints, and the 2-metric.

Sec.(9) concludes the paper with a concise summary.

Throughout our development, we take the point of view that all steps should be motivated from the principle of scale invariance, even though the structures which emerge are already familiar from quantum systems. Thus, scale-invariance provides an underlying reason, within the context of a spacetime manifold, for the physical existence and importance of Hilbert space, harmonic operators and the mode and Virasoro algebras which is independent of the insights of string theory and quantum theory. We use the name *conformal dynamics* to describe this new point of view.

2 The String Algebra

We first provide a brief reminder of some basic features of string and the Virasoro algebra. Following [1], the construction of the algebra of string begins with the classical mode expansion. For simplicity we will restrict our remarks to open string, since the results for closed string are similar and readily available elsewhere [1].

The classical mode expansion for string, after imposing open string boundary conditions, is

$$X^\mu(\sigma, \tau) = x^\mu + l^2 p^\mu \tau + il \sum_{n>0} \frac{1}{n} \, \alpha_n^\mu \cos n\sigma \; e^{-in\tau} \tag{2}$$

where the length parameter l is related to the string tension by $l = (\pi T)^{-1/2}$. Here x^μ is the center of mass of the string and p^μ is the total linear momentum. The angular momentum of the string may be written in terms of x^μ, p^μ and the remaining mode amplitudes α_n^μ as

$$M_{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu) \tag{3}$$

Notice that eq.(2) does *not* have independent left- and right-moving modes, but only supports the wave modes of the form

$$e^{im(x-y)} + e^{-im(x+y)} \tag{4}$$

Similarly, the closed string boundary conditions restrict the solution to even modes. We shall see that this counting is important for understanding string from the point of view of scale invariance.

Also from the mode amplitudes, we can construct the quantities,

$$L_m = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \tag{5}$$

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which form the Virasoro algebra

$$[L_m, L_n]_{P.B.} = (m - n)L_{m+n} \tag{6}$$

where $[L_m, L_n]_{P.B.}$ denotes the Poisson bracket with respect to the set of mode coordinates $(x^\mu, p^\mu, \alpha_n^\mu)$. To quantize the string, a Hilbert space is introduced and the mode amplitudes α_n^μ become Hilbert space operators $\hat{\alpha}_n^\mu$, $\hat{\alpha}_0^\mu = l\hat{p}^\mu$. These operators comprise a countable set of creation and anihilation operators for modes of vibration of the string. Also, central charges, $c_{mn} = \frac{D}{12}(m^3 - m)\delta_{m+n}^0$, may be added to the quantum commutators because the operators $\hat{\alpha}_m^\mu$ are defined only up to a phase, that is, they form a projective representation of the string mode algebra. The quantum string algebra is therefore

$$[\hat{L}_m, \hat{L}_n] = (m - n)\hat{L}_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n}^0 \tag{7}$$

Quantization also leads to a normal ordering ambiguity in the definition of $\hat{L}_0$, which is conveniently defined as

$$\hat{L}_0 \equiv \frac{1}{2}\hat{\alpha}_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} \hat{\alpha}_{-n}\hat{\alpha}_n \tag{8}$$

We can extend the Virasoro algebra further by including symmetries generated by the stress-energy tensor of the string. The extension takes the form

$$[\hat{M}^{\alpha\beta}, \hat{P}_n^\gamma] = \eta^{\beta\gamma}\hat{P}_n^\alpha - \eta^{\alpha\gamma}\hat{P}_n^\beta \tag{9}$$

$$[\hat{P}_m^\alpha, \hat{P}_n^\nu] = 0 \tag{10}$$

$$[\hat{P}_m^\alpha, \hat{L}_n] = (m - n)\hat{P}_{m+n}^\alpha \tag{11}$$

A similar extension of the Virasoro symmetry follows from the symmetry of the tangent tower, by considering inhomogeneous linear maps [2], though we will not go into the details of this extension here.

Finally, the physical space of states $|\psi\rangle$ is given by the conditions:

$$\hat{L}_m |\psi\rangle = 0 \qquad \forall m > 0 \tag{12}$$

$$(\hat{L}_0 - a) |\psi\rangle = 0 \tag{13}$$

These conditions guarantee that physical states have positive norm, and therefore form a Hilbert space.

3   The symmetry of scale invariant theories

We now begin our demonstration that Hilbert space, acted on by mode amplitudes and the Virasoro algebra *including the central charge*, occurs *classically* as part of the tangent

space structure to any scale-invariant theory of gravity. The mode and Virasoro algebras occur as the maximally and minimally commuting, complete Lie algebras of linear transformations of the Hilbert space sector of the tangent space.

To provide the concreteness of a typical background context, we give the structure equations for the simplest class of scale-invariant geometries. The only property of these geometries of interest to us here is the tower structure of the tangent space. Many other geometries besides the class presented here (for example, conformal or biconformal geometries) have the same tangent space tower structure.

As the example context, consider a d-dim pseudo-Riemannian geometry with curvature 2-form $\mathbf{R}_b^a$, vielbein 1-form $\mathbf{e}^a$, and spin connection 1-form ω_b^a, satisfying

$$\mathbf{de}^a = \mathbf{e}^b \wedge \omega_b^a \tag{14}$$

$$\mathbf{R}_b^a = \mathbf{d}\omega_b^a + \omega_d^a \wedge \omega_b^d \tag{15}$$

where $a, b = 0, 1, \ldots d - 1$. The vielbein provides an orthonormal basis, so that Latin indices are contracted using η_{ab}. Conversion between orthonormal and general coordinate bases is accomplished using the vielbein component matrix, $e_\mu{}^a$ and its inverse.

We can make this geometry scale-invariant by introducing a gauge 1-form θ (the "Weyl vector") for local scale changes together with its curvature 2-form

$$\Omega = \mathbf{d}\theta \tag{16}$$

Unlike Weyl's theory, this curvature is related to the Hamiltonian structure of fields on spacetime [3]. In particular, Ω does *not* represent the electromagnetic field.

The metric is given in terms of the components of the vielbein by

$$g_{\alpha\beta} = e_\alpha{}^a e_\beta{}^b \eta_{ab} \tag{17}$$

where η_{ab} is the d-dim Minkowski metric. When a local change in the choice of units changes $g_{\alpha\beta}$ by a (conventional) factor $\exp(2\varphi)$, we (conventionally) take the vielbein to scale by $\exp(\varphi)$

$$\mathbf{e}^a \longrightarrow \exp(\varphi)\, \mathbf{e}^a \tag{18}$$

The metric is said to have a scale weight of two, and the vielbein a scale weight of one. In addition, we must modify eq.(14) to give the dependence of the connection on θ. We have

$$\mathbf{de}^a = \mathbf{e}^b \wedge \omega_b^a + \mathbf{e}^a \wedge \theta \tag{19}$$

Eqs.(15,16 & 19) describe scale invariant (or Weyl) geometry. We will not develop these equations further here, but instead turn our attention to the tangent space V at a point P of the Weyl, or other scale-invariant, geometry.

A priori, the vectors in V transform under the Weyl group, i.e., Lorentz transformations and scalings. Translations may also be allowed, depending on the model. However,

the situation is more complicated than this. The generator of scale changes, D, called the dilation operator, may have different eigenvalues when acting on different physical fields. These eigenvalues, called the *conformal weights* of the fields, give rise to equivalence classes of vectors. In Sec.(6) we show that there are actually two different equivalence relations consistent with the weight tower structure, with distinct physical consequences. In each case, the total tangent vector space at a point P is the direct sum of these equivalence classes.

Thus, V is a space of vectors $v_{(w)}{}^a$, where in addition the Lorentz index a, each vector carries a weight $w \in W$ which characterizes its transformation under scalings. Closure of the symmetry algebra under commutation requires the weight set W to be closed under the addition of any two *different* elements of W. In addition to the integers, J, the rationals, Q, and the reals, R, this constraint permits the finite sets $\{0, 1\}$ and $\{-1, 0, 1\}$. For the remainder of our discussion, we will assume[1] that the weight set is J.

When the vielbein is scaled by $\exp \varphi$, a vector field of weight n scales by $\exp n\varphi$:

$$v_{(n)}^a \longrightarrow e^{n\varphi} \, v_{(n)}^a \tag{20}$$

Infinitesimally, we have

$$D v_{(n)}^a = n v_{(n)}^a \tag{21}$$

where D is the generator of dilations. Thus, the covariant derivative of a weight-n vector field is

$$D_\mu v_{(n)}^a = \partial_\mu v_{(n)}^a - v_{(n)}^b \omega_{\mu b}{}^a + v_{(n)}^a n\theta_\mu \tag{22}$$

where θ_μ is the Weyl vector. Notice how there is effectively a different gauge vector, $n\theta_\mu$, for each weight of vector field. We handle this multiplicity more explicitly and efficiently by introducing the tangent tower. The *tangent tower* is the direct sum of vector spaces $V_{(n)}$ of each allowed weight $n \in J$

$$V = \bigoplus_{n \in J} V_{(n)} \tag{23}$$

Now consider homogeneous[2] linear maps on V. These maps must preserve the direct sum structure of V, and therefore must map vectors of definite weight to other vectors of definite weight.

$$v_{(n)}^a{}' = \Delta_{(n,m)}{}^a{}_b \, v_{(m)}^b \tag{24}$$

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[1] Almost all physical systems allow the choice $W = J$. To arrive at a set of integer weights W for physical fields, we may use the following procedure. First, using the fundamental constants, express the units of each given physical field as a power of length. The set S of all such powers for a given field theory is typically a set of rational numbers with a least common denominator, r. The set of realized physical weights is then $rS \subseteq J$ and we can take the integers J as the weight set. In this case, the weight of any given physical field is r times the power of length associated with that field. Only if S has no least common denominator must the set of weights be extended to (at least) the set of rationals, Q.

[2] As mentioned in Sec.(2), the *inhomogeneous* maps on V lead to the momentum extension of the Virasoro algebra given by eqs.(7-9).

Here the labels (n, m) on $\Delta_{(mn)}{}^a{}_b$ are just a mnemonic to tell us that $\Delta_{(mn)}{}^a{}_b$ carries the weight-m vector to a weight-n vector.

The form of $\Delta_{(mn)}{}^a{}_b$ is restricted by requiring consistency with the usual procedures for combining physical fields, which let us construct vector fields of all weights from a single vector field of weight n. This is simple, for given a nonvanishing vector field[3] $v^a_{(n)}$ of weight n, and the metric of weight 2, the nonvanishing unit weight scalar field

$$\phi_{(1)} \equiv \left(\eta_{ab} v^a_{(n)} v^b_{(n)} \right)^{1/(2n+2)} \tag{25}$$

can be used multiplicatively to produce vector fields of arbitrary weight,

$$v^a_{(n+k)} = (\phi_{(1)})^k v^a_{(n)}. \tag{26}$$

Clearly, $v^a_{(n+k)}$ and $v^a_{(n)}$ are parallel. Moreover, given this construction, it is clear that at any point the angle between $v^a_{(n)}$ and a second similar set of vector fields $w^a_{(n)}$, is the same as that between $v^a_{(n+k)}$ and $w^a_{(n+k)}$ independent of the weight n and the value of k. Therefore, the Lorentz transformations contained in $\Delta_{(mn)}{}^a{}_b$ must be independent of weight, and $\Delta_{(mn)}{}^a{}_b$ is a direct product

$$\Delta_{(mn)}{}^a{}_b = \Delta_{(nm)} \Lambda^a{}_b \tag{27}$$

where $\Lambda^a{}_b$ is a Lorentz transformation and $\Delta_{(nm)}$ acts only on the weight indices.

Now consider $\mathcal{T} : V^n \to R$, the space of multilinear maps over $V = \oplus V^n$, typified by sums over a multi-vector basis,

$$T^{ab...c}_{n_1+n_2+\cdots n_3} = \sum u_{(n_1)}{}^a v_{(n_2)}{}^b \cdots w_{(n_3)}{}^c \tag{28}$$

Though $\mathcal{T}$ seems to be the space of rank-n tensors over V, eq.(28) is actually too restricted. Because of the separation of Lorentz transformations and weight transformations, we can factor apart the direction and weight properties of the vectors, and use the factors to construct tensors of arbitrarily mixed type. The Lorentz direction factor will be an element of d-dim, projective Minkowski space, $PM^d \equiv M^d/R^+$. The weight factor, $\Delta_{(mn)}$, acts on the integers, J, so we consider tensors of type (r, s) to be linear maps $T^{(r,s)} : (PM^d)^r \otimes J^s \to R$. Such a tensor, of rank (r, s), will have components of the form

$$T^{a_1 a_2 \cdots a_r}_{n_1 n_2 \cdots n_s} \tag{29}$$

where in contrast to eq.(28), we may have $r \neq s$. For example, the $(0, s)$ tensor $T_{n_1 \cdots n_s}$ is a multilinear map on an s-tuple of rank $(0, 1)$ fields, $(\varphi_{(k_1)}, \ldots, \varphi_{(k_s)})$. Rank $(0, 1)$ fields are Lorentz scalars and weight tower vectors. Tower tensors of rank (r, s) must not be confused with tensors of mixed covariant and contravariant types. Once the metric is

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[3] Any time orientable spacetime manifold has a nonvanishing vector field.

introduced, both the Lorentz rank r, and the weight rank s, may be subdivided among covariant or contravariant types.

It is important to distinguish the conformal weight of a tensor from its rank over weight space. Each (r,s) tensor $T^{(r,s)}$ acts on s fields of definite or indefinite weight. If we evaluate $T^{(r,s)}$ on a set of fields $\phi_1, \cdots, \phi_s$ of definite weights $n_1, \cdots, n_s$ respectively, then $T^{(r,s)}$ is of weight k if the weight of $T^{(r,s)}(\phi_1, \cdots, \phi_s)$ is $k + n_1 + \cdots + n_s$, independent of the values of $n_1, \cdots, n_s$. Such a tensor $T^{(r,s)}$ has tower rank s and conformal weight k. It is possible for $T^{(r,s)}$ to have no weight, or to have a weight only with respect to certain of its indices.

Our principal concerns here will be with Lorentz tensors of weight ranks $s = 0, 1$ and 2. In many instances the Lorentz rank is immaterial, though it is often simplest to consider Lorentz scalars or vectors, $r = 0, 1$. We now consider each of these values of s in turn.

Tensors of type $(r,0)$ are rank-r Lorentz tensors, $T^{a_1 a_2 \cdots a_r} = T_0^{a_1 a_2 \cdots a_r}$ which are annihilated by the dilation operator

$$DT^{a_1 a_2 \cdots a_r} = 0 \tag{30}$$

Of these, the Lorentz scalars are centrally important. This is because *type* $(0,0)$ *tensors are the only physically measurable quantities*, all others changing under conformal transformations, Lorentz transformations, or both. Tensors with $s \neq 0$ are useful, however, for the construction of weight zero tensors, just as Lorentz tensors are useful for constructing Lorentz scalars.

Type $(0,1)$ tensors are weighted Lorentz scalars such as $\phi_{(1)}$ defined in eq.(25), or their sums. To express higher weight objects we can use powers of $\phi_{(1)}$, such as

$$\phi_{(k)} \equiv \left(\phi_{(1)} \right)^k \tag{31}$$

as an example of a scalar with weight k, i.e., $D\phi_{(k)} = k\phi_{(k)}$. Taking a fixed set $\{\phi_{(k)}\}$ as a *vector* basis for the type $(0,1)$ tensors, we see that the general element

$$f(\phi) = \sum_{k=0}^{\infty} \beta_k \phi_{(k)} \tag{32}$$

may be written as an arbitrary convergent power series in ϕ, of indefinite weight. Linear combinations of fields of different scale weights are unphysical in themselves, but may nonetheless provide measurable quantities. For example, while $f(\phi)$ is not measurable directly, the 0-weight scalar

$$\sum_{k=0}^{\infty} \beta_k \beta_{-k} \tag{33}$$

constructible from $f(\phi)$ can in principle be measured. In Sec.(5), the allowed class of such indefinite weight superpositions will be defined.

The tensors of type $(1,1)$ are simply the vectors v_n^a we began with, together with their linear combinations. Again, the idea of taking linear combinations of lengths and

areas seems physically odd, but as long as we remember that only type $(0,0)$ objects are measurable, there is no inherent problem.

Finally, we will be particularly interested in objects of type $(r, 2)$ because they act as linear maps on the weight labels of type $(r', 1)$ objects. Our first result below gives a set of necessary and sufficient conditions for the class of type $(0, 2)$ objects, called *weight maps,* to reduce to the Virasoro algebra, while Theorem 2 similarly characterizes the mode algebra. The type $(1, 2)$ objects are vector-valued weight maps including a subset analogous to the quantized Fourier coefficients $\hat{\alpha}_n^a$ of strings. We explore the properties of these in detail in Sec.(8).

We now begin our study of weight maps. Consider possible maps on the set of integer labels $\mathcal{M} : J \longrightarrow J$, that is, type $(0, 2)$ tensors. Together with the generators of the Weyl group, we demand that these operators have the following properties:

1. The operators form a Lie algebra.

2. All operators have definite weight.

3. The algebra is maximally non-commuting, i.e., $[M_{(m)}, M_{(n)}] \neq 0$ for all $m \neq n$.

4. The algebra is complete, containing exactly one operator of each allowed weight.

Assumptions (1) and (2) are basic physical constraints. The demand for an algebra instead of a group follows from the basic commutator relation $[D, M_{(n)}] = nM_{(n)}$, which shows that definite weight operators extend the Lorentz-plus-dilation Lie algebra. Assumption (3) avoids trivial subgroups of operators and isomorphic representations.[4] Though it is clearly possible to find algebras with multiple operators or no operators of a given weight, condition (4) is a reasonable completeness demand.

The remainder of Sec.(3) and all of Sec.(4) together provide the proof of the following central result. Sec.(3) includes the more general elements of the proof, as well as two additional theorems, while Sec.(4) is devoted to the detailed inductive analysis required by the proof of Theorem 1.

Theorem 1 *The unique maximally non-commuting, complete Lie algebra of definite-weight operators is the Virasoro algebra.*

To make the flow of the argument clearer, we choose a particular matrix basis for the next steps. While the matrix basis we use in this section provides an intuitive handle on the algebra, other representations generalize readily to other rank objects, allow easier calculation and reveal new properties of the objects. Therefore, in Sec.(5) we will introduce an alternative representation.

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[4]For example, the set of even integers is an allowed weight set isomorphic to J.

To construct the matrix basis, we pick an arbitrary definite-weight vector $v^a_{(n)}$ in V. By the construction above, we can produce an infinite tower of parallel vectors of each weight $m \in J$. Because Lorentz transformations decouple from weight transforms, we need only consider maps which take some vector $v^a_{(n)}$ in this tower to some other $v^a_{(m)}$ in the tower. Therefore, we can suppress the Lorentz index and choose a basis in which the components of $v^a_{(n)} = v_{(n)}$ are given by

$$(v^a_{(n)})^i = v^a \delta^i_n \tag{34}$$

or simply

$$(v_{(n)})^i = \delta^i_n \tag{35}$$

Alternatively, we can work with $(0,1)$ tensors, with a definite-weight basis $\phi_{(n)}$, and avoid Lorentz structure altogether. In this case, the basis is essentially the same, $(\phi_{(n)})^i = \delta^i_n$, but there is no suppressed index[5].

Now consider those linear maps that associate to each definite-weight vector another definite-weight vector. We can describe this class as follows. Let

$$\Phi = \{\phi \mid \phi : J \to J\} \tag{36}$$

be the automorphism group of the integers. Then the weight tower is preserved by the set of maps

$$\mathcal{M}^\Phi = \{M^{\phi,\alpha} \mid M^{\phi,\alpha} v_{(k)} = \alpha^\phi_k v_{\phi(k)}, \phi \in \Phi\} \tag{37}$$

where for each $M^{\phi,\alpha}$ the set $\{\alpha^\phi_k \mid k \in J, \alpha^\phi_k \in R\}$ may be chosen arbitrarily.

The action of D on the set $\mathcal{M}^\Phi$ is not well-defined because there is no weight associated with a general map M^ϕ. For this reason, and because well-defined weight is necessary for constructing conformal scalars, we restrict our attention to maps of definite weight. Definite-weight maps are defined by using the additivity property of conformal weights. Consider the action of D on a mapping $M^\phi v_{(k)}$ of a weight-k vector

$$DM^\phi v_{(k)} = [D, M^\phi] v_{(k)} + M^\phi D v_{(k)} \tag{38}$$

Solving for the commutator and using $M^\phi v_{(k)} = \alpha_k v_{(\phi(k))}$ results in

$$[D, M^\phi] v_{(k)} = (\phi(k) - k) \alpha_k v_{(\phi(k))} = (\phi(k) - k) M^\phi v_{(k)} \tag{39}$$

The definite-weight weight maps are those for which this is an operator relation,

$$[D, M^\phi] = (\phi(k) - k) M^\phi \tag{40}$$

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[5]Ultimately, it makes sense to identify all of the ranks $(1,0)$, $(1,1)$ and $(0,1)$ as vectors in V, with $(1,0)$ vectors being those that lie on a single "floor" of the tower and $(0,1)$ vectors having the zero Lorentz vector as their projection to any floor. Although we do not introduce a combined basis notation covering all types, we will refer to all three types as vectors.

which requires that the coefficient $(\phi(k) - k)$ be independent of k. Since $\phi(k) \in J$, this means there exists some integer n such that $\phi(k) = n + k$ so the maps ϕ are restricted to the countable set

$$\Phi_N = \{\phi_n \mid \phi_n(k) = n + k\} \subset \Phi \tag{41}$$

where $n \equiv \phi(0)$. Therefore, we can denote the set of definite-weight operators by

$$\mathcal{M}^{N,A} = \{M^{n,\alpha} \mid M^{n,\alpha} v_{(k)} = \alpha_k^n v_{(n+k)}\} \tag{42}$$

and replace eq.(40) by

$$[D, M^{n,\alpha}] = n M^{n,\alpha}. \tag{43}$$

Since $M^{n,\alpha}$ acts on any definite-weight vector $v_{(k)}$ according to

$$M^{n,\alpha} v_{(k)} = \alpha_k^n v_{(n+k)} \tag{44}$$

the components $(M^{n,\alpha})^i{}_j$ of $M^{n,\alpha}$ in the definite-weight basis are

$$(M^{n,\alpha})^i{}_j = \sum_{s=-\infty}^{\infty} \alpha_s^n \delta^i_{(s+n)} \delta_j^{(s)} \tag{45}$$

Now we ask for a set consisting of exactly one operator $M_{(n)}$ of each weight. Then a single J-tuple of components α_s^n for each $n \in J$ characterizes the full set. We therefore drop the extraneous α label, writing $M_{(n)}$ in place of $M^{n,\alpha}$.

Next, consider the Lie algebra requirement. In order for the set of operators $M_{(n)}$ to form a Lie algebra $\mathcal{L}^N$ they must close under commutation and satisfy the Jacobi identity. But the Jacobi identity involving $M_{(m)}, M_{(n)}$, and D gives immediately

$$[D, [M_{(m)}, M_{(n)}]] = (m + n)[M_{(m)}, M_{(n)}] \tag{46}$$

so that the commutator must be of weight $m + n$, and therefore proportional to $M_{(m+n)}$:

$$[M_{(m)}, M_{(n)}] = \sum_k c^k{}_{mn} M_{(k)} \tag{47}$$

$$= \sum_k c_{mn} \delta^k_{m+n} M_{(k)} = c_{mn} M_{(m+n)}. \tag{48}$$

Since eq.(45) provides a matrix representation for these operators (albeit doubly infinite) the Jacobi identities between Ms follow automatically. The commutators with Poincaré generators are well-known once we establish that $\mathcal{L}_N$ is the Virasoro algebra, so the only condition left to impose is closure under commutation. The commutator has components

$$([M_{(m)}, M_{(n)}])^i{}_k = \sum_{s=-\infty}^{\infty} (\alpha_{s+n}^m \alpha_s^n - \alpha_s^m \alpha_{s+m}^n) \delta^i_{(s+m+n)} \delta_k^{(s)} \tag{49}$$

$$= c_{mn}(M_{(m+n)})^i{}_k = c_{mn} \sum_{s=-\infty}^{\infty} \alpha_s^{m+n} \delta^i_{(s+m+n)} \delta_k^{(s)} \tag{50}$$

The algebra closes if and only if, for structure constants c_{mn} and all s, m and n the α_s^n satisfy the *primary recursion* relation

$$\alpha_{s+n}^m \alpha_s^n - \alpha_s^m \alpha_{s+m}^n = c_{mn} \alpha_s^{m+n} \tag{51}$$

which provides a strong constraint since the m, n and s dependence on the left must factor into the product of a term depending only on m and n, and a term depending only on s and the sum $m + n$.

We conclude this section with an aside, before completing the proof of Theorem 1 in Sec.(4). Here, we solve the primary recursion relation for the case of the maximally commuting, rather than the maximally non-commuting, complete Lie algebra of definite weight operators. The calculations of Sec.(4) are parallel to these, but require substantially more effort.

Consider a set of operators $\mathcal{J} = \{J_n\}$ where J_n has components

$$(J_n)^i{}_j = \sum_{s=-\infty}^{\infty} \alpha_s^n \delta_{n+s}^i \delta_j^s \tag{52}$$

in the definite-weight basis. In the maximally commuting case, $c_{mn} = 0$, and the primary recursion reduces to

$$\alpha_{s+n}^m \alpha_s^n - \alpha_s^m \alpha_{s+m}^n = 0 \tag{53}$$

Setting $n = 0$ gives $\alpha_s^0 = const$. Then, we can normalize our set of basis vectors so that

$$J_1 v_{(k)} = v_{(k+1)} \tag{54}$$

i.e., $\alpha_s^1 = 1$. Next, setting $n = 1$ in eq.(53) shows the constancy of all of the coefficients with a given value of m,

$$\alpha_{s+1}^m = \alpha_s^m \tag{55}$$

This leaves only one overall constant for each J_m, which can be absorbed by redefining the operators. We therefore have the immediate solution algebra, $\mathcal{J}$, of unit off-diagonal matrices

$$(J_m)^i{}_j \equiv \sum_{s=-\infty}^{\infty} \delta_{s+m}^i \delta_j^s \tag{56}$$

which have weight m, commute with one another, and satisfy

$$J_m J_n = J_{m+n} \tag{57}$$
$$J_m v_n = v_{m+n} \tag{58}$$

When the projective structure of the tower is used to allow central charges, the algebra $\mathcal{J}$ becomes the mode algebra. The vector-valued extension of $\mathcal{J}$ figures importantly in Sec.(8), in our discussion of the relationships between rank $(1, 2)$ tensors and string.

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Notice that the action of J_m does not distinguish among the different vectors v_m, but has a "spectrum" with $\lambda_s = 1$ for all s (see eq.(44)). This can be changed by adjoining to each operator in $\mathcal{J}$ the operator D :

$$L_m \equiv J_m D \tag{59}$$

The resulting algebra, $\mathcal{J}D$ is isomorphic to the Virasoro algebra, as is readily shown by computing

$$[L_m, L_n] = [DJ_m, DJ_n] = (n-m)J_{m+n}D = (n-m)L_{m+n} \tag{60}$$

It also follows immediately that the spectrum of values α_s^m arising from the action of $J_{(m)}D$ on $v_{(s)}$ is nondegenerate for each m.

We therefore have proved the following two results:

Theorem 2 *The maximally commuting, complete Lie algebra of definite-weight operators is isomorphic to the algebra $\mathcal{J}$ of unit, off-diagonal matrices.*

Theorem 3 *The algebra $\mathcal{J}D$ is isomorphic to the classical Virasoro algebra.*

In Sec.(4) we complete the proof of Theorem 1 by carrying out steps similar to those in the proof of Theorem 2 above, but assuming that c_{mn} is nonzero whenever $m \neq n$. This condition is sufficient to produce the Virasoro algebra directly.

4   Solution of the primary recursion

We now turn to the consequences of the primary recursion relation

$$\alpha_{s+n}^m \alpha_s^n - \alpha_s^m \alpha_{s+m}^n = c_{mn}\alpha_s^{m+n} \tag{61}$$

for possible forms of $M_{(n)}$.

Before invoking eq.(61), we note that some of the coefficients α_s^m and structure constants c_{mn} are already determined while others can be fixed by rescaling the operators $M_{(m)}$. First, α_s^m and c_{mn} can be related if we begin in the adjoint representation, where from eq.(47)

$$\left(M_{(k)}\right)^i{}_j = c^i{}_{kj} = c_{kj}\delta^i_{(k+j)} \tag{62}$$

If eq.(45) is to agree with the adjoint expression we must have

$$c_{kj}\delta^i_{(k+j)} = \sum_{s=-\infty}^{\infty} \alpha_s^k \delta^i_{(s+k)} \delta^{(s)}_j = \alpha_j^k \delta^i_{(j+k)} \tag{63}$$

or

$$c_{kj} = \alpha_j^k \tag{64}$$

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from which we immediately see that $\alpha_j^k = -\alpha_k^j$ and in particular, $\alpha_k^k = 0$ for all k.

Next, since the unique operator of weight zero must be the dilation generator D, (since D acts on weight indices only) we have

$$Dv_{(m)} = mv_{(m)} \tag{65}$$

from which it follows that

$$\alpha_m^0 = m \tag{66}$$

while eq.(43) or eq.(64) shows that

$$c_{0m} = m \tag{67}$$

Moving away from the adjoint representation, we now can adjust the structure constants $c_{-1,m}$ by rescaling each $M_{(m)}$ by a factor β_m. Such rescaling does not alter $\alpha_m^m = 0$. Then from eq.(50)

$$c_{-1,m} \equiv c_{-1,m}^{new} = \frac{\beta_{m-1}}{\beta_{-1}\beta_m} c_{-1,m}^{old} \tag{68}$$

Because c_{mn} is antisymmetric, $c_{-1,-1}$ vanishes. Choosing the βs so that

$$\beta_m = \frac{c_{-1,m}^{old}}{(m+1)\beta_{-1}}\beta_{m-1} \qquad \forall m \geq 0 \tag{69}$$

$$\beta_{m-1} = (m+1)\frac{\beta_{-1}}{c_{-1,m}^{old}}\beta_m \qquad \forall m \leq -2 \tag{70}$$

fixes the remaining $c_{-1,m}$ so that

$$c_{-1,m} = m+1 = -c_{m,-1} \qquad \forall m \tag{71}$$

Note that since the commutator of two distinct operators never vanishes, we cannot have $c_{-1,m} = 0$ for $m \neq -1$. Eqs.(69) and (70) fix all but two of the β_m: we still may freely fix any one coefficient in the set $\{\beta_{-1}, \beta_1, \beta_2, \beta_3 \ldots\}$ and any second coefficient in the set $\{\beta_{-2}, \beta_{-3}, \beta_{-4} \ldots\}$.

Finally, we can redefine the basis $v_{(m)}$ in terms of the action of either $M_{(1)}$ or $M_{(-1)}$. That is, it is always possible to replace $v_{(m)}$ by $\eta_m v_{(m)}$ and choose the constants η_m so that

$$M_{(-1)}v_{(m)} = (m+1)v_{(m-1)} \tag{72}$$

This normalization of $v_{(m)}$ again scales the values of the α_s^m in eq.(45), but has no other effect. The values of α_m^{-1} given by eq.(72)

$$\alpha_m^{-1} = m+1 \tag{73}$$

are consistent with eq.(61).

These normalizations together with eq.(61) are sufficient to determine all of the remaining α_s^m. The proof begins by setting $n = -1$ and $m = 1$ in eq.(61) (since the equation is an identity if either $m = 0$ or $n = 0$). Using eq.(73), this gives

$$(s+1)\alpha_{s-1}^1 - (s+2)\alpha_s^1 = -2s \tag{74}$$

and induction readily yields $\alpha_s^1 = s - 1$ for all $s \neq -2$. The value $\alpha_{-2}^1 = -3$ may be fixed by the choice of β_1, so we have

$$\alpha_s^1 = s - 1, \qquad \forall s \tag{75}$$

Collecting our results so far, we have

$$\alpha_s^1 = s - 1 \tag{76}$$
$$\alpha_s^0 = s, \qquad c_{0s} = s \tag{77}$$
$$\alpha_s^{-1} = s + 1, \qquad c_{-1,s} = s + 1 \tag{78}$$

for all s, where we are still free to fix any one element of the set $\{\beta_{-2}, \beta_{-3}, \beta_{-4} \ldots\}$.

Since $M_{(m)}$ with $m \in \{-1, 0, 1\}$ form a closed subalgebra, we do not yet have sufficient seed values to use eq.(61) recursively. Therefore, we next consider eq.(61) for three cases: $(n = -1, m = 2), (n = 1, m = -2)$ and $(n = 2, m = -2)$. The first gives

$$(s+3)\alpha_s^2 - (s+1)\alpha_{s-1}^2 = 3(s-1) \tag{79}$$

which determines

$$\alpha_s^2 = s - 2 \qquad \forall s \neq -2, -3 \tag{80}$$
$$\alpha_{-2}^2 + \alpha_{-3}^2 = -9 \tag{81}$$

The second case is similar except that $c_{-2,1}$ is unknown:

$$(s-1)\alpha_{s+1}^{-2} - (s-3)\alpha_s^{-2} = (s+1)c_{-2,1} \tag{82}$$

Eq.(82) permits α_s^{-2} to be found in terms of $c_{-2,1}$ for all $s < 2$ and $s > 3$, and provides one remaining relation

$$\alpha_3^{-2} + \alpha_2^{-2} = 3c_{-2,1} \tag{83}$$

Finally, the $(n = 2, m = -2)$ case,

$$\alpha_{s+2}^{-2}\alpha_s^2 - \alpha_s^{-2}\alpha_{s-2}^2 = sc_{-2,2} \tag{84}$$

has useful special cases. For $s = 4$ we find $\alpha_6^{-2} = 2c_{-2,2}$ while for $s = 2$ we have $\alpha_2^{-2} = c_{-2,2}$. From the expression for α_6^{-2} in terms of $c_{-2,1}$ already determined from eq.(82) it follows that

$$4c_{-2,1} = 3c_{-2,2} \tag{85}$$

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Therefore, using β_{-2} to set $\alpha_2^{-2} = 4$ immediately yields $c_{-2,2} = 4$ and $c_{-2,1} = 3$. With these structure constants determined, the remaining values, $\alpha_s^{-2} = s \mp 2$ follow quickly for all s.

Now two induction arguments give the remaining α_s^m. For $m < -2$ we use $n = -1$ in eq.(61) with $\alpha_s^{-2} = s + 2$ as the seed value for the induction. Substituting for known values and rearranging

$$\alpha_s^{m-1} = \frac{1}{(m+1)} \left((s+m+1)\alpha_s^m - (s+1)\alpha_{s-1}^m \right) \tag{86}$$

gives the expected values $\alpha_s^m = s - m$ for all $m < -2$ and all s.

To accomplish the same result for positive m requires eq.(61) with $n = 1$, but now we also need values for c_{1m}. There are several steps needed to find c_{1m}. First set $n = -1$ and $s = -1$ in eq.(61) to find α_{-1}^m for positive m. The resulting values, $\alpha_{-1}^m = -(m+1)$, together with the $(n = -1, s = 0)$ equation give $\alpha_0^m = -m$. Then setting $n = -1$ and $s = 1$ yields $\alpha_1^m = 1 - m$. Finally, using these values of α_0^m and α_1^m in eq.(61) with $n = +1$ produces the necessary result, $c_{m1} = 1 - m$. With these values for the structure constants, the $n = +1$ equation becomes

$$\alpha_s^{m+1} = \frac{1}{(m-1)} \left((s+m-1)\alpha_s^m - (s-1)\alpha_{s+1}^m \right) \tag{87}$$

and induction based on $\alpha_s^2 = s - 2$ works as expected.

Combining everything, we have simply

$$\alpha_s^m = s - m = c_{ms}, \qquad \forall m, \forall s \tag{88}$$

completing the proof that conditions $(1 - 4)$ imply the Virasoro form for the operators $M_{(m)}$. This concludes the proof of Theorem 1.

Before moving on to alternate representations for weight maps in the next section, we note that the commuting algebra $\mathcal{J}$ makes it possible to shift α_s^m by an arbitrary amount without changing the Virasoro commutators. Let

$$M_{(k)}^\alpha = M_{(k)} + \alpha J_{(k)} \tag{89}$$

so that

$$(M_{(k)}^\alpha)^i{}_j = \sum_{s=-\infty}^{\infty} (s - m + \alpha)\delta_{s+m}^i \delta_j^s \tag{90}$$

for an arbitrary constant α. It is straightforward to check that $M_{(k)}^\alpha$ satisfies

$$[M_{(k)}^\alpha, M_{(l)}^\alpha] = (l - k)M_{(k+l)}^\alpha \tag{91}$$

for any α. In particular, the choice $\alpha = m$ simplifies the representation to the form

$$(M_{(k)}^{\alpha=m})^i{}_j = \sum_{s=-\infty}^{\infty} s\delta_{s+m}^i \delta_j^s \tag{92}$$

which we will use in Sec.(5).

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5   Alternative representations of weight maps

Many calculations on the infinite tower of weight states are greatly simplified by the use of a continuous basis, which we define below. Essentially, we use the infinite dimensional $(0,1)$ vectors as the coefficients in a Fourier series. Then a matrix operator on the $(0,1)$ vector space may be written as a double Fourier series. The operator becomes a function of two variables, a fact which lets us represent vector-valued operators as strings in Sec.(7). The product of two operators is given by integration over the "inner" pair of variables while the restriction to definite weight operators leads ultimately to wave solutions. In this section we develop these ideas, keeping as our primary motivation the representation of scale-invariant operators in $\mathcal{L}^N$.

Given what we already know, we can find motivation for moving to a Fourier representation in the shifted sum between α_{s+n}^m and β_s^n of the multiplication rule

$$(M_{(m)}N_{(n)})^i{}_j = \sum_{s=-\infty}^{\infty} \alpha_{s+n}^m \beta_s^n \delta_{(s+m+n)}^i \delta_{(s)j} \tag{93}$$

Noting that a similar shifted sum occurs when multiplying Fourier transforms, we can try to reformulate the multiplication rule in terms of Fourier series. It turns out that the simplest anzatz works: we can model the maps by treating their components as the coefficients of a double Fourier series, that is, we define a distribution of two variables in terms of the components, $(M)^m{}_n$ of an arbitrary operator. However, having noted this clue that a Fourier representation will work, it is most logical to begin the formal development with the continuous representation of $(0,1)$ vectors and build up from there.

Therefore, we start with the definition of the function space on which the Virasoro operators act, using the components of $(0,1)$ vectors in the definite weight basis as coefficients in a Fourier series. Beginning with the definite weight basis $(v_{(k)})^m = \delta_k^m$ we find that the basis functions

$$f_k(x) = \sum_{m=-\infty}^{\infty} (v_k)^m e^{imx} = e^{ikx} \tag{94}$$

where $x \in [-\pi, \pi]$, are individual Fourier modes.[6] For a general linear combination[7] of

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[6]Certain choices have been made in writing eq.(94). We could equally well take the real part of this and subsequent expressions so that $f : [-\pi, \pi] \to \mathcal{R}$. Also, notice that the choice of the region $[-\pi, \pi]$ is arbitrary. Not only could we pick any other bounded interval, but we could use a circle instead so that $f : \mathcal{S}^1 \to \mathcal{R}$. The only strict requirement is the use of a compact region, since compactness guarantees the existence of Fourier series for piecewise continuous functions. Even this requirement may be relaxed if we allow the index set to be R rather than J, permitting the use of a Fourier integral representation on the real line (and consequently, an "infinite length string" representation). It is important to remember as we look at different representations that the only required property is the representation of weight maps.

[7]Recall that indefinite weight combinations cannot be measured directly. For a crude example, it is as if we were to make a vector of position, momentum and force, $v^\mu = (\ldots, v_{-2}^\mu, v_{-1}^\mu, v_0^\mu, v_1^\mu, \ldots) = (\ldots, F^\mu, p^\mu, 0, x^\mu, \ldots)$ and look at the dynamics of the entire object at once. Nonetheless, as long as we restrict any physical predictions to states of zero weight, there is no reason indefinite vectors v cannot

the basis vectors

$$(v)^m = \sum_{k=-\infty}^{\infty} \alpha^k (v_k)^m \tag{95}$$

we can write the Fourier series

$$f(x) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \alpha^k (v_k)^m e^{imx} = \sum_{k=-\infty}^{\infty} \alpha^k e^{ikx} \tag{96}$$

but it is clear that we need (in either representation) some convergence criterion. Such a criterion is provided most easily by a norm.

Choosing a suitable norm for the functions $f(x)$ is not as simple as it seems. Setting $g(x) = \sum_{k=-\infty}^{\infty} \beta^k e^{ikx}$ and allowing the coefficients α^k, β^k to be complex (as they are, for example, if $f(x^\mu; x)$ and $g(x^\mu; x)$ below are complex scalar fields on spacetime) the usual Hilbert space norm,

$$(f, g) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(x) g(x) dx = \sum_{k=-\infty}^{\infty} (\alpha^k)^* \beta^k \tag{97}$$

is unstable under conformal transformations. To see this, note that under a scale change e^φ, the function $f(x)$ becomes

$$f'(x) = \sum_{k=-\infty}^{\infty} \alpha^k e^{k\varphi} e^{ikx} \tag{98}$$

so that

$$(f', f') = \sum_{k=-\infty}^{\infty} (\alpha^k)^* \alpha^k e^{2k\varphi} \tag{99}$$

which is only guaranteed to converge for series $\{\alpha^k\}$ having greater than exponential convergence. This constrains the function space quite severely (although it is consistent with the Gaussian envelopes of typical wave packets). It is more natural and satisfactory to begin with a scale invariant norm.

To implement scale invariance of the norm we first introduce the *conformal conjugate* of a vector $g(x) = \sum_{k=-\infty}^{\infty} \beta^k e^{ikx}$ by

$$\bar{g}(x) \equiv \sum_{k=-\infty}^{\infty} \beta^{-k} e^{ikx} \tag{100}$$

It is convenient to used raised and lowered indices in the usual way, writing $\beta^k \to \beta_k \equiv \beta^{-k}$. Then a conformally invariant inner product may be written using the summation convention

$$f \cdot g = \alpha^m \beta_m = \alpha_m \beta^m \tag{101}$$

be used. In particular, notice that the whole of v^μ transforms as a Lorentz vector, consistent with the decoupling of Lorentz and weight transformations.

This inner product is the natural weight tower extension of the Killing metric of the conformal group. When the coefficients β_k are real, conformal and complex conjugation are equivalent

$$\bar{g}(x) = \sum_{k=-\infty}^{\infty} \beta_k e^{ikx} = \sum_{k=-\infty}^{\infty} \beta^{-k} e^{ikx} = \sum_{m=-\infty}^{\infty} \beta^m e^{-imx} = g^*(x) \tag{102}$$

Returning to the norm, we impose *both* complex and conformal conjugation.

$$\langle f, g \rangle \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \bar{f}^*(x) g(x) dx = \sum_{m=-\infty}^{\infty} \alpha_m^* \beta^m = \langle g, f \rangle^* \tag{103}$$

This norm is real but not positive definite, since

$$\langle f, f \rangle = \sum_{m=-\infty}^{\infty} \alpha_m^* \alpha^m \tag{104}$$

$$= \alpha_0^* \alpha_0 + \sum_{m=1}^{\infty} [\alpha^{-m*} \alpha^m + \alpha^{m*} \alpha^{-m}] \tag{105}$$

is linear, not quadratic, in α^m. However, there are many regions of the space of functions $f(x)$ for which this expression has definite sign. Most of these depend on careful matching of phases, but two classes are easier to describe and a third deserves mention.

Consider the class of functions f with Fourier coefficients in the definite weight basis satisfying $\alpha_k = \lambda_k \alpha^k$ for all $k > 0$ and arbitrary numbers $\lambda_k > 0$. For such functions (which are gauge-equivalent to the class of self-dual functions, $\bar{f}(x) = f(x)$, identical to the symmetric functions), the norm $\langle f, g \rangle$ is

$$\langle f, f \rangle = \sum_{m=-\infty}^{\infty} \lambda_m |\alpha^m|^2 \tag{106}$$

which is positive definite and vanishes if and only if $f = 0$. Unfortunately the presence of λ_k again destablilzes the class, since only certain sequences $\{\lambda_k\}$ give convergence. We might consider restricting $\lambda_k \leq 1$, but then a function f with $\alpha_k = \lambda_k \alpha^k$ will have a conformal dual with $\alpha_k = \frac{1}{\lambda_k} \alpha^k$, and therefore not in the space unless $\lambda_k = 1$. We are therefore constrained to the self dual functions. Convergence in the self-dual norm gives a Hilbert space, $\mathcal{H}$. This result is of considerable importance, since it shows that *there is a Hilbert space of conformally self-dual vectors automatically associated with the tangent space at each point of any scale-invariant geometry.*

A second class of functions, those which are anti-self-dual, are similarly shown to form a space with negative definite norm.

A third class of functions of interest are those for which $\alpha_k = 0$ for all $k \geq 0$. Any such function f has only positive frequency modes in its Fourier expansion

$$f(x) = \sum_{k=1}^{\infty} \alpha^k e^{ikx} \tag{107}$$

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Consequently, all are null, $\langle f, f \rangle = 0$ since the conformal conjugate vanishes. Of course, the presence of null vectors prevents our using this norm to define a Hilbert space for these functions, but now the usual Hilbert norm (f, g) is stable for *contractive* gauge changes.

We now move from the continuous representation of vectors to consider the continuous representation of weight maps. Writing maps as double Fourier series with components in the definite weight basis, $f_k(x)$, the maps become functions of two variables, $(x, y) \in \mathcal{N} \equiv [-\pi, \pi] \times [-\pi, \pi]$. Since the weight maps M of previous sections take $(0, 1)$ vectors to $(0, 1)$ vectors, the inner product requires one index in each position $(M)^m{}_n$. A generic weight map therefore takes the form

$$M(x, y) \equiv \sum_{m,n} (M)^m{}_n e^{imx} e^{-iny} \tag{108}$$

The product of two operators follows from eq.(103) as

$$MN \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} M(x, z) N(z, y) dz \tag{109}$$

$$= (2\pi)^{-1} \int \sum_{m,n} (M)^m{}_n \sum_{p,q} (N)^p{}_q \exp imx \exp -i(n-p)z \exp -iqy \tag{110}$$

$$= \sum_{m,q} (\sum_n (M)^m{}_n (N)^n{}_q) \exp imx \exp -iqy \tag{111}$$

$$= (MN)(x, y) \tag{112}$$

which shows that the sign convention of eq.(108) and the definition of the product via eq.(109) are in agreement with the matrix representation.

It is illustrative to use the 2-dim representation to find the commutator of a general operator with D, and to rederive the generic form, eq.(45), of a definite weight operator. The convention for indices introduced above (which ties together complex and conformal conjugation) requires us to write

$$D(x, y) = (2\pi)^{-1} \sum_{m,n} (D)^m{}_n e^{imx} e^{-iny} \tag{113}$$

$$= (2\pi)^{-1} \sum_{m,n} \sum_s s \delta_s^m \delta_n^s e^{imx} e^{-iny} \tag{114}$$

$$= (2\pi)^{-1} \sum_s s e^{is(x-y)} \tag{115}$$

$$= -i \frac{\partial}{\partial x} \delta(x - y) \tag{116}$$

We now show that this simple form for $D(x, y)$ leads to a set of definite weight functions $f_k(x)$ with only pure modes $f_k(x) \sim e^{ikx}$, and to a set of definite weight operators $M_{(k)}(x, y)$ with only "right moving modes", $M_{(k)}(x, y) \sim M_{(k)}(x - y)$. While some such restriction on operators is a necessary concomitant of definite-weight tensors, "left-moving

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modes" are present in all but the definite-weight basis. In cases where both left- and right-moving modes appear, they are not independent but are determined by a single analytic function. We show below that operators with truly independent left- and right-moving modes can be constructed using operators of definite *anti*-weight.

To begin, consider the action of $D(x, y)$ on a $(0, 1)$ vector

$$f(x) = \sum_{k=-\infty}^{\infty} c^k e^{ikx} \tag{117}$$

Note that regardless of the values of the c^k we have $f(\pi) = f(-\pi)$. The condition for $f(x)$ to be of definite weight is

$$Df = \lambda f \tag{118}$$

or, in terms of eq.(109)

$$Df \equiv \int_{-\pi}^{\pi} D(x, y) f(y) dy \tag{119}$$

Unfortunately, the integration by parts used to evaluate eq.(119) gives a surface term of the form

$$i\delta(x - \pi) f(\pi) - i\delta(x + \pi) f(-\pi) \tag{120}$$

which vanishes only if $f(\pi) = f(-\pi) = 0$. In combination with the eigenvalue relation, eq.(118), $f(^+_-\pi) = 0$ is inconsistent.

Therefore, in order to get a complete set of weighted functions satisfying eq.(118) we must include a singular correction term in the definition of D. The most general term which can be used to cancel terms of the form of eq.(120) is

$$
\begin{aligned}
c(x, y) &= c_1\delta(x + \pi)\delta(y + \pi) + c_2\delta(x + \pi)\delta(y - \pi) \\
&\quad + c_3\delta(x - \pi)\delta(y + \pi) + c_4\delta(x - \pi)\delta(y - \pi)
\end{aligned} \tag{121}
$$

We now show that the distribution $c(x, y)$ can be chosen to eliminate all surface terms, whether we apply D from the right or from the left. Moreover, the choice for $c(x, y)$ works the same way for the action of D on any rank of tensor.

To find $c(x, y)$, let D be given by

$$D(x, y) = -i\frac{\partial}{\partial x}\delta(x - y) - ic(x, y) \tag{122}$$

with $c(x, y)$ given by eq.(121), and consider the right and left action of D on the rth slot of a rank $(0, k)$ tensor, that is,

$$(DT)_r \equiv \int_{-\pi}^{\pi} dz D(x, z) T(x_1, x_2, \ldots, x_{r-1}, z, x_{r-2}, \ldots, x_k) \tag{123}$$

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and

$$(TD)_r \equiv \int\limits_{-\pi}^{\pi} dz\, T(x_1, x_2, \ldots, x_{r-1}, z, x_{r-2}, \ldots, x_k) D(z, x) \tag{124}$$

where

$$T(x_1, x_2, \ldots, z, \ldots, x_k) = \sum T^{n_1 \ldots n \ldots n_k} e^{i n_1 x_1 + \ldots + i n z + \ldots + i n_k x_k} \tag{125}$$

For $(DT)_r$ the integration by parts gives a surface term of the form

$$
\begin{aligned}
S.T. &= -i\left[-\delta(x - z) T(x_1, x_2, \ldots, z, \ldots, x_k)\right]_{-\pi}^{\pi} \\
&\quad -i \int\limits_{-\pi}^{\pi} c(x, z) T(x_1, x_2, \ldots, z, \ldots, x_k) \tag{126}
\end{aligned}
$$

which cancels provided

$$
\begin{aligned}
c_1 + c_2 &= 1 \tag{127} \\
c_3 + c_4 &= -1 \tag{128}
\end{aligned}
$$

The surface term for $(TD)_r$ also vanishes, provided

$$
\begin{aligned}
c_1 + c_3 &= -1 \tag{129} \\
c_2 + c_4 &= 1 \tag{130}
\end{aligned}
$$

Both conditions are satisfied if D is of the form

$$D(x, y) = -i[\partial_x \delta(x - y) - \delta(x - \pi)\delta(y + \pi) + \delta(x + \pi)\delta(y - \pi)] \tag{131}$$

plus an arbitrary multiple of the purely surface term

$$
\begin{aligned}
E &= [\delta(x - \pi)\delta(y - \pi) - \delta(x - \pi)\delta(y + \pi) \tag{132} \\
&\quad -\delta(x + \pi)\delta(y - \pi) + \delta(x + \pi)\delta(y + \pi)] \tag{133}
\end{aligned}
$$

Since $ET = TE = 0$ for any tensor T (as long as $T(\pi) = T(-\pi)$) we can simply drop the E term.

Now that the surface terms vanish, the left and right action of $D(x, y)$ on the rth slot of any tensor is simply given by the integrated part, [8]

$$
\begin{aligned}
(DT)_r &= -i\frac{\partial T}{\partial x_r} \tag{134} \\
(TD)_r &= i\frac{\partial T}{\partial x_r} \tag{135}
\end{aligned}
$$

---

[8]In evaluating $\int f(y)\delta(x - y)$ we have taken the full value of the δ-functions at the endpoints $\pm\pi$. Other choices can be made by adjusting the surface term.

Using eqs.(134) and (135) we now find a complete set of eigenfuntions and definite weight operators.

Substituting $D(x, y)$ into eq.(118), we find the differential equation

$$Df^{(k)} = -i\frac{\partial f^{(k)}(x)}{\partial x} = kf^{(k)}(x) \tag{136}$$

which is solved by the set of pure mode functions

$$f^{(k)}(x) = A_k e^{ikx} \tag{137}$$

Thus, just like plane waves in quantum mechanics, the simplest set of eigenfunctions of $D(x, y)$ do not lie in the Hilbert space. It is possible to develop a self-dual basis, although $D(x, y)$ then takes a more complicated form.

For rank two tensors (or equivalently rank $(r, 2)$ for any r) we can compute the commutator,

$$[D, M](x, y) = (DM)_1 - (MD)_2 \tag{138}$$

which allows us to define operators $M_{(k)}$ of definite weight by demanding

$$[D, M_{(k)}] = kM_{(k)} \tag{139}$$

Substituting from eqs.(134) and (135),

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) M(x, y) = ikM(x, y) \tag{140}$$

with the immediate solution:

$$M_{(k)}(x, y) = e^{ikx} h_k(x - y) = e^{ikx} \sum_{m=-\infty}^{\infty} \alpha_m^k e^{im(x-y)} \tag{141}$$

where h is an arbitrary function, which we take to be analytic. Notice that $M_{(k)}(x, y)$ has only "right-moving" modes.

It is natural enough to wonder whether varying the treatment of the continuous representation above could lead to "left-moving" modes as well, in the expression for $M_{(k)}(x, y)$. As mentioned at the start of the Section, it is indeed possible to choose a different basis than the pure mode $f_{(k)}(x)$ used here, such that the result for $M_{(k)}(x, y)$ is a superposition of right and left modes. As one might expect, the amplitudes of the two modes are still expressed in terms of a *single* function $h(z)$, rather than two independent functions $f_1(x + y)$ and $f_2(x - y)$. The full expressions for $D, M_{(k)}$, and $f_{(k)}$ in a set of alternative bases are derived in Appendix 1.

A superposition of *independent* left and right moving modes arises only if we consider operators which have definite weight under *anti*-commutation with the dilation generator,

$$\{D, N_{(k)}\} \equiv DN_{(k)} + N_{(k)}D = kN_{(k)} \tag{142}$$

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Eqs.(134) and (135) lead immediatly to

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) N_{(k)}(x,y) = ik N_{(k)}(x,y) \tag{143}$$

with the solution

$$N_{(k)}(x,y) = e^{ikx} g_k(x+y) \tag{144}$$

Any operator satisfying eq.(142) will be said to have *anti-weight k*. Notice that the distinct behavior of "right-moving" and "left-moving" modes displayed by $M_{(k)}(x,y)$ and $N_{(k)}(x,y)$ is a form of heterosis (see [1], pg 306). We stress that this heterosis is distinct from that of the heterotic string. For string, the asymmetry is due to the assignment of bosonic and fermionic operator types to the left and right moving mode amplitudes, and does not result in anticommutation with the dilation operator.

The operators $N_{(k)}$, which anticommute with D, are just as useful for constructing zero-weight action functionals as the $M_{(k)}$ because such action functionals will generally be quadratic (or of even powers) in operator-valued fields. Thus, in a term in an action such as

$$S \sim \int N_{(k)} N_{(k)} \tag{145}$$

the quadratic expression $N_{(k)} N_{(k)}$ commutes with D just as the corresponding expression $M_{(k)} M_{(-k)}$ does.

It is easy to check using anticommutation with D, that the products $M_m N_n$ and $N_n M_m$ have weights $n+m$ and $n-m$, respectively. Interestingly, the products $N_m N_n$ and $N_n N_m$ also have different weights, $m-n$ and $n-m$. This leads to complicated commutation relations for the N_m. However, the use of t'Hooft commutators restores simplicity. Working with,

$$N_m(x,y) = e^{ikx} \sum_{m=-\infty}^{\infty} m e^{im(x+y)} \tag{146}$$

we define the t'Hooft commutator as

$$[N_m, N_n]_{tH} \equiv N_m N_n - e^{2i(m-n)x} N_n N_m \tag{147}$$

Then the extra phase equalizes the weights and we find

$$[N_m, N_n]_{tH} = (m-n) M_{m-n} \tag{148}$$

Similarly, the t'Hooft commutator of M_m with N_n is proportional to N_{m+n}, so the full set $\{M_m, N_n\}$ form a quantum group with modular symmetry.

The most general linear combination of definite weight operators, including both commuting and anticommuting types, is

$$M(x, y) \;=\; \sum_{k=-\infty}^{\infty} \left[\lambda_k M_{(k)}(x, y) + \eta_k N_{(k)}(x, y) \right] \tag{149}$$

$$\;=\; \sum_{k=-\infty}^{\infty} e^{ikx} \sum_{m=-\infty}^{\infty} \left[\alpha_m^k e^{im(x-y)} + \beta_m^k e^{-im(x+y)} \right] \tag{150}$$

where some of the operator properties, including the commutation and anticommutation with D, are contained in the (x, y)-dependence of $M(x, y)$. This is an important notational difference from standard quantum field theory expressions, in which the *amplitudes* are the operators. Here, the amplitudes are simply numbers. In fact, the operator content of these expressions is distributed, because it is the amplitudes that govern *which* algebra the operators represent. For example, in the expression above we might choose $\alpha_m^k = m, \beta_m^k = 0$ so that $M(x, y)$ lies in the Virasoro algebra, $\mathcal{L}^N$, or we could set $\alpha_m^k = 1, \beta_m^k = 0$ so that $M(x, y)$ is an element of the mode algebra, $\mathcal{J}$. This distributed character of the operator properties complicates the comparison with standard results.

The matrix results of the preceeding sections may also be found using the continuous representation. Substitution of the Fourier series for $M_{(k)}(x, y)$ shows the required agreement with eq.(45). It is also straightforward to verify the convolution form of eq.(93) for the product of two operators, as well as the condition for closure under commutation, eq.(61).

In the specific case of the Virasoro operators, for which we can take $\alpha_s^k = s$, the distributional character of the Virasoro operators in $\mathcal{L}_N$ may be expressed directly:

$$M_{(k)}(x, y) \;=\; (2\pi)^{-1} \sum_{m,n} \sum_{s=-\infty}^{\infty} s \delta_{s+k}^m \delta_n^s e^{imx} e^{-iny} \tag{151}$$

$$\;=\; (2\pi)^{-1} \sum_n n e^{ikx} e^{in(x-y)}$$

$$\;=\; (2\pi)^{-1} e^{ikx} (-i\frac{\partial}{\partial x}) \sum_n e^{in(x-y)}$$

$$\;=\; -i e^{ikx} \frac{\partial}{\partial x} \delta(x - y)$$

where we have inserted an overall factor of $(2\pi)^{-1}$ in the normalization for convenience. Unfortunately, this distributional form for $M_{(k)}$ leads to surface terms when $M_{(k)}$ acts on arbitrary tensor fields. While it may be possible to find corrective additional terms as we did for $D(x, y)$, it is simpler to eliminate the surface terms by going to a derivative representation. To find the derivative representation, we easily check that the integral action of $M_{(k)}(x, y)$ on the mth weight slot of an arbitrary rank (r, s) tensor $T^{\mu \cdots \nu}(x_1, \ldots x_s)$ may be written as

$$M_{(k)}T = \int dz M_{(k)}(x, z) T(\cdots, z, \cdots) = L_k(x) T(\cdots, x, \cdots) + surface\ terms \tag{152}$$

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where L_k is the operator

$$L_k \equiv -ie^{ikx}\frac{\partial}{\partial x} \tag{153}$$

The generators L_k are immediately recognizable as the generators of the diffeomorphism group of the projective punctured plane, or equivalently, the diffeomorphisms of S^1, provided we interpret x as an angular variable. It is easy to check that

$$[L_m, L_n] = (n - m)L_{m+n} \tag{154}$$

The action of L_k on a vector $f(x)$ is

$$\begin{aligned} L_k f(x) &= -ie^{ikx}\frac{\partial}{\partial x} \quad v \sum_{m=-\infty}^{\infty} \beta_m e^{imx} \\ &= v \sum_{m=-\infty}^{\infty} m\beta_m e^{i(k+m)x} \end{aligned} \tag{155}$$

and when f reduces to a particular mode f_m

$$L_k f_m(x) = mv \exp i(k + m)x = mf_{m+k} \tag{156}$$

as expected.

These calculations may be repeated for the commuting algebra $\mathcal{J}$. As noted following eq.(150), it is only the choice of α_s^n in eq.(150) that fixes the algebra to be $\mathcal{J}$ instead of $\mathcal{L}^N$. We immediately see that $\mathcal{J}$ may be represented by the operators

$$J_n = e^{inx}\delta(x - y) \tag{157}$$

with the identity operator given by $J_0 = \delta(x - y)$. Then the product of two Js is simply

$$J_n J_m = J_{n+m} \tag{158}$$

so that

$$[J_n, J_m] = 0 \tag{159}$$

By analogy with the definition of L_k we can define operators H_m satisfying

$$\begin{aligned} (JT)(x_1, \cdots, x, \cdots, x_s) &= \int dz J(x, z)T(\cdots, z, \cdots) \tag{160} \\ &= H_m(x)T(\cdots, x, \cdots) + surface\ terms \tag{161} \end{aligned}$$

Clearly, $H_m = e^{imx}$ acts through multiplication alone. Once again we see the relationship between $\mathcal{J}$ and $\mathcal{L}_N$, since $L_n = H_n D$. Quite generally, for any weight-n operator $M_{(n)}$, the product $H_m M_n$ is a corresponding operator with weight $n + m$.

Again, we define a "left-moving" set of operators, $K_{(k)}$, which anticommute with D

$$\{D, K_m\} = mK_m \tag{162}$$

but this time have vanishing t'Hooft commutator

$$[K_m, K_n]_{tH} \equiv K_m K_n - e^{2i(m-n)x} K_n K_m = 0 \tag{163}$$

K_m therefore takes the form

$$K_n = e^{inx} \delta(x + y) \tag{164}$$

in the continuous representation. The corresponding integrated operator, I_m inverts the sign of the tensor field it acts on, so it must be written as the product of a phase and a parity operator. Let

$$(PT)_r(x_r) = T(-x_r) \tag{165}$$

for the rth slot of any rank tensor T. Then $I_m = e^{imx}P$.

The full collection of operators $M_{(k)}, N_{(k)}, J_{(k)}$ and $K_{(k)}$ together form a t'Hooft-Virasoro quantum algebra. The full algebra is given in Appendix 2.

In subsequent sections we will employ the mixed representation,

$$M = \sum \alpha_s^k e^{is(x-y)} \tag{166}$$

in addition to the matrix and continuous representations used so far. This allows greater generality. Often it is convenient to let the mixed and continuous representations differ by a factor of 2π, for example by writing D as $-i\partial_x \delta(x - y)$ in the continuous representation but as $\sum s e^{is(x-y)}$ in the mixed representation. The extra constant factor generally causes no difficulty.

To conclude this section, we note how neatly the complex, continuous representation reflects the weight tower structure. Phase transformations act to change the weights of fields, while a differential representation of the Virasoro algebra emerges in a natural way from the noncommuting transformations. Further details on the relationship between units, length scales and the continuous representation by phases are given in Appendix 3.

We now turn to an examination the tangent space, including a look at the equivalence relations which define the tangent tower, and a study of the central charges which arise as a result of the projective character of each of the "floors" of the weight tower.

6 Projective structure

While the full algebra of transformations $\mathcal{M}$ acts on the weighted tangent spaces of a scale-invariant geometry, not all of these transformations are physically meaningful because parallel vectors are equivalent up to a change of gauge. Since physical laws must be gauge invariant, those laws must be unchanged if they are rewritten in terms of any different representative of a given projective equivalence class. This is why the weight tower is defined using projective Minkowski spaces as the floors of the tower. In fact, the whole projective tower structure defines an equivalence relation: any two Lorentz-parallel

vectors of the same weight are equivalent. In this section, we turn this last statement around. We will look at a heirarchy of three equivalence relations, and the implications each has for tangent structure, operator equivalence, and Lie algebras of operators.

To define these three equivalence relations, we must first characterize the full tangent space, V, in a way unbiased regarding the presence of a tower structure. We therefore assume that V has a countable, definite weight basis and that each definite weight subspace is d-dimensional. While weaker assumptions are possible, this one guarantees that we know how to evaluate D on any vector, and is consistent with the physical models we are trying to describe. Given the definite weight basis $\{w_m, e_{m\mu}{}^a\}$, we can construct definite weight projection operators, either as the outer product of basis vectors such as

$$P_m = w_m \otimes w_m \tag{167}$$

or more formally using D,

$$P_m = \prod_{k \neq m} \frac{D - k}{m - k} \tag{168}$$

Given such a basis and a complete set of projection operators, we can examine equivalence relations on V.

The hierarchy of equivalence relations arises from different ways of specifying when two vectors are parallel. The weakest relation defines two vectors as equivalent if they are Lorentz-parallel. Thus, two vectors v_1 and v_2 in the tangent space at a point P are *Weyl equivalent*, $v_1 \cong_{Weyl} v_2$, if for every integer m and in each gauge there exists a number $\lambda_{m,\varphi} \neq 0$ such that

$$P_m v_1 = \lambda_{m,\varphi} P_m v_2 \tag{169}$$

The numbers $\lambda_{m,\varphi}$ may be both gauge- and m-dependent. Weyl equivalence recognizes only the different Lorentz directions with no regard for weight, so the V modulo Weyl equivalence is simply projective d-dim Minkowski space. Weyl equivalence makes no distinction between parallel vectors of different scaling weights. As a result, the tower structure is not present.

Strengthening Weyl equivalence slightly, we define v_1 and v_2 to be *weakly equivalent*,

$$v_1 \cong_W v_2 \tag{170}$$

if for every integer m there exists a nonzero number λ_m such that

$$P_m v_1 = \lambda_m P_m v_2 \tag{171}$$

where λ_m may depend on the weight m, but must be independent of the gauge. This will be the case if there exists a nondegenerate gauge-invariant $(0, 2)$ tensor Λ such that

$$v_1 = \Lambda v_2 \tag{172}$$

Λ can be gauge-invariant only if it is of weight zero. Therefore, by eq.(45) it must have components

$$(\Lambda)^n{}_m = \lambda_n \delta^n{}_m \tag{173}$$

as may also be seen directly from the invariance of eq.(172). The eigenvalues, λ_n may be arbitrary nonzero numbers, possibly different for each weight in the tower, so that the set of allowed Λs forms a rather large group, $(R - \{0\})^J$. Nonetheless, two definite-weight vectors of different weights are never weakly equivalent, so weak equivalence is sufficient to define a tower structure. When restricted to vectors of definite weight, weak equivalence clearly implies Weyl equivalence, though Weyl equivalence is not defined for more general tower vectors.

The tower structure arising from weak equivalence is extremely simple. Computing the quotient, V modulo weak equivalence, we find that two $(0, 1)$ tensors are equivalent if and only if they have the same *distribution* of nonvanishing components. Every equivalence class of such weight vectors contains one representative with all of its components equal to either zero or one. For this reason, while weak equivalence is sufficiently strong to recognize the tower structure, it will turn out to strongly limit possible algebras of weight maps.

Finally, two vectors $v_1, v_2,$ are *strongly equivalent,* $v_1 \cong v_2$ if for every integer m there exists a nonzero number λ such that

$$P_m v_1 = \lambda P_m v_2 \tag{174}$$

where λ is independent of both the weight and the gauge. Stated more simply, v_1 and v_2 must be parallel, so there exists a gauge constant $\lambda \neq 0$ such that $v_1 = \lambda v_2$. The gauge-invariant λ is just the measurable ratio of the magnitudes of v_1 and v_2. Notice that two vectors of different, definite weight can never be strongly equivalent, so strong equivalence, like weak equivalence, defines a tower structure. Correspondingly, the ratio of magnitudes of two vectors of different definite weight does not give a measurable ratio. In fact, strong equivalence defines the tower structure of the preceeding sections. The definition requires the vectors to be parallel both in spacetime direction and in their tower direction. Therefore, if the components of v_1 and v_2 are $(v_1)_m^\mu$ and $(v_2)_m^\mu$, respectively, then $(v_1)_m^\mu = \lambda (v_2)_m^\mu$ for all μ and all m. Note that indefinite weight vectors *can* be strongly equivalent, and that the subspace of vectors of definite weight, modulo strong equivalence, forms a projective Minkowski space.

Strong equivalence implies weak equivalence, since whenever $v_1 = \lambda v_2$, we can choose Λ to be the diagonal matrix with all λ_n equal to λ so that $\Lambda = \lambda 1$. Conversely, two vectors which are weakly equivalent and such that $\Lambda = \lambda 1$ for some λ, are strongly equivalent.

Next, we consider the effect of these equivalence relations on weight maps. Since Weyl equivalence does not give a tower structure, it also allows no weight maps and has no

algebraic structure beyond the Weyl group. Therefore, for the remainder of the section, we restrict our attention to strong and weak equivalence.

While both the weak and strong equivalence classes of equal weight parallel vectors preserve the tower structure, they lead to very different equivalence classes. Nonetheless, for definite-weight vectors, there is no difference between the two - any two vectors which are Lorentz parallel and of the same weight are both strongly and weakly equivalent. The difference shows up only for weight superpositions. Consider two superpostions containing the same distribution of floors of the weight tower such that the projection to any particular floor gives Lorentz-parallel vectors. Then the two superpositions will in general be weakly, but not strongly, equivalent.

For example, consider two Lorentz vectors $u_{(n)}$ and $v_{(m)}$ of weights n and m, respectively. Then the superpositions $au_{(n)} + bv_{(m)}$ and $cu_{(n)} + dv_{(m)}$ are weakly equivalent for any nonvanishing a, b, c and d, but are strongly equivalent only if the 2-vectors (a, b) and (c, d) are parallel. In the weak case, the 0-weight weight map

$$\Lambda = \begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & & \frac{a}{c} & & & \\ & & & \frac{b}{d} & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix} \tag{175}$$

accomplishes the equivalence.

The differences between weak and strong equivalences are even more pronounced when we consider transformations. We define two weight-n transformations $M_{(n)}$ and $N_{(n)}$ to be strongly or weakly equivalent, $M_{(n)} \cong N_{(n)}$ or $M_{(n)} \cong_W N_{(n)}$ if the vectors $M_{(n)}v$ and $N_{(n)}v'$ are strongly or weakly equivalent whenever v and v' are correspondingly equivalent. It is trivial to show that strongly equivalent maps must be proportional, $M = \lambda N$, but weakly equivalent maps are highly degenerate.

The definition of weakly equivalent maps requires a nondegenerate zero-weight map Λ satisfying

$$M_{(n)}v = \Lambda N_{(n)}v' = \Lambda N_{(n)}\Lambda_1 v \tag{176}$$

for all v and all Λ_1, or simply

$$M_{(n)} = \Lambda N_{(n)}\Lambda_1 \tag{177}$$

In components this implies the existence of numbers λ_k, λ_{1k} such that

$$(M_{(n)})^k{}_m = (\Lambda)^k{}_l (N_{(n)})^l{}_i (\Lambda_1)^i{}_m = \lambda_k \lambda_{1m} (N_{(n)})^k{}_m \tag{178}$$

Since $M_{(n)}$ and $N_{(n)}$ are weight-n maps,

$$(M_{(n)})^k{}_m = \sum_{s=-\infty}^{\infty} \alpha_s^{(n)} \delta_{s+n}^k \delta_m^s = \lambda_k \lambda_{1m} (N_{(n)})^k{}_m = \sum_{s=-\infty}^{\infty} \beta_s^{(n)} \lambda_k \lambda_{1m} \delta_{s+n}^k \delta_m^s \tag{179}$$

resulting in

$$\alpha_s^{(n)} = \beta_s^{(n)} \lambda_{s+n} \lambda_{1s} \tag{180}$$

for all s. This expression provides a different λ_{s+n} for each value of $\beta_s^{(n)} \lambda_{1s}$. Since n is fixed, and the operators $M^{(n)}$ and $N^{(n)}$ are determined by $\alpha_s^{(n)}$ and $\beta_s^{(n)}$, *every two operators of the same (definite) weight having the same zeros are weakly equivalent.* Consequently, since the matrix representation of a definite weight operator is a shifted-diagonal matrix, there is only one operator with a given shift and a given set of zeros, up to weak equivalence.

The difference between strong and weak equivalence is especially important when we consider Lie algebras of equivalence classes of operators. For the moment, let $\cong$ denote either equivalence relation and set $\mathcal{S} \equiv \{ M^\alpha \mid \alpha \in A \}$ where each M^α represents an equivalence class $M^\alpha = \{ M^\alpha_\beta \}$ of operators modulo $\cong$, with β indexing the members of each class. Then $\mathcal{S}$ will be a Lie algebra consistent with $\cong$ provided

$$[M^\alpha, M^{\alpha'}] \cong c_{\alpha\alpha'\alpha''} M^{\alpha''} \tag{181}$$

that is,

$$[M^\alpha_{\beta_1}, M^{\alpha'}_{\beta_2}] \cong c_{\alpha\alpha'\alpha''} M^{\alpha''}_{\beta_3} \tag{182}$$

for any choices of the representatives β_1, β_2 and β_3. If we pick initial representatives $M^\alpha_0, M^{\alpha'}_0$ and $M^{\alpha''}_0$ such that

$$[M^\alpha_0, M^{\alpha'}_0] = c_{\alpha\alpha'\alpha''} M^{\alpha''}_0 \tag{183}$$

then there must exist some Λ' in the equivalence group such that

$$[\Lambda_1 M^\alpha_0, \Lambda_2 M^{\alpha'}_0] = c_{\alpha\alpha'\alpha''} \Lambda' \Lambda_3 M^{\alpha''}_0 \tag{184}$$

for all Λ_1, Λ_2 and Λ_3. Setting $\Lambda' = \Lambda(\Lambda_3)^{-1}$, we require existence of Λ such that

$$[\Lambda_1 M^\alpha_0, \Lambda_2 M^{\alpha'}_0] = c_{\alpha\alpha'\alpha''} \Lambda M^{\alpha''}_0 \tag{185}$$

for each choice of Λ_1 and Λ_2. For strong equivalence, this is obviously possible because each $\Lambda = \lambda 1$ commutes with any choices of the operators M^α_0 and we simply set $\lambda = \lambda_1 \lambda_2$. In this case, we have the algebras described in Sec.(2), including $\mathcal{L}_N$ and $\mathcal{J}$. However, for weak equivalence eq.(185) is highly constraining.

A sufficient condition[9] for a Lie algebra of operators to satisfy eq.(185) for weak equivalence is to demand that the Λ_1 and Λ_2 operators factor out to the left in the same way as for strong equivalence, but this requires

$$[M^\alpha_0, \Lambda_2] = 0 \tag{186}$$

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[9]The potentially weaker assumption that $\exists \Lambda^\alpha_2$ such that $M^\alpha \Lambda_2 = \Lambda^\alpha_2 M^\alpha$ can also be shown to be inconsistent with an algebra of definite weight operators $M^{(k)}$ with $k \neq 0$.

The only set of operators which commute with all zero-weight matrices Λ_2 are the zero-weight matrices themselves. We therefore consider the physical implications of the zero-weight algebra.

The algebra (or group) of diagonal matrices is composed of commuting observables. We may therefore conjecture that they correspond to classical variables, and that classical physics rests ultimately on the assumption of weak equivalence. Physically, weak equivalence corresponds to ignoring any physical information contained in the multiplicative constants of weight superpositions. It only matters *which* weights are superimposed, not *how much* of each is present.

By contrast, strong equivalence gives meaning to the relative constants between definite weight components, just as mixed quantum states depend on the relative phases between the pure states of which they are composed. The operator algebra projectively preserved by the strong equivalence relation is not directly measurable, since not all operators are of zero weight. However, it is possible to measure combinations of these operators or their effects, for example by using the Hilbert norm discussed above.

If this interpretation of weak and strong equivalences as determining classical and quantum operators is correct, then the difference between classical and quantum physics lies in whether the ultimate dynamical laws apply to each floor of the tower independently, or to arbitrary-weight vector superpositions as a whole. If each conformal weight evolves independently of the others, then the operator structure arising from weak equivalence will be enough to account for the evolution, so the motion will be described by classical (i.e., commuting) operators. But if tower superpositions evolve in a way that reflects relationships between different conformal weight objects, then strong equivalence and the resulting non-commuting operators must play a role, and the system will be a more complicated one. Strong conformal dynamics is described in Sec.(7), where we show that strongly equivalent parallel transport obeys the Schrödinger equation.

Next we consider an extremely important consequence of projective equivalence - central charges. In string theory, it is equivalence under unitary projection that gives the Virasoro algebra its now famous central charges. The unitary equivalence occurs because the theory has been quantized. However, *scaling geometries already provide real projective equivalence without quantization.* We now move to a demonstration that the same central charges occur. For the remainder of our discussion, we will assume the tower structure to be determined by strong equivalence.

Following [4], suppose we have a Lie group G which acts on a space, with elements $a, b, ... \in G$. Projective equivalence of physical states on that space (whether scale or unitary) allows us to use a projective representation H of G, with $A, B, ... \in H$, such that

$$A(a)B(b) = \lambda C(ab) \tag{187}$$

where λ is a scale or phase factor. This is a necessary condition for the Lie algebras of G

and H to differ by the addition of central charges to the commutation relations in H,

$$[H_a, H_b] = c_{ab}^d H_d + c_{ab}1 \tag{188}$$

while sufficiency depends on whether the central term can be removed by redefining the generators H_a. In eq.(188) the central charges $c_{ab} = -c_{ba}$ must satisfy the Jacobi-like identity [4]

$$c_{bc}^a c_{ad} + c_{cd}^a c_{ab} + c_{db}^a c_{ac} = 0 \tag{189}$$

These relations hold whether the factor λ in eq.(187) is real or complex.

For the Virasoro algebra, $c_{mn}^k = (m-n)\delta_{m+n}^k$, so the central charges must satisfy

$$(m-n)c_{m+n,k} + (k-m)c_{m+k,n} + (n-k)c_{k+n,m} = 0 \tag{190}$$

Now, eq.(188) is invariant under the replacement

$$\hat{c}_{mn} = c_{mn} + (m-n)\phi_{m+n} \tag{191}$$

for any ϕ_k. This means that it is possible to remove some of the potential central charges by a shift in the generators. In the present case, the commutator of $M_{(k)}$ with D,

$$[M_{(k)}, D] = -kM_{(k)} + c_{k0}1 \tag{192}$$

becomes

$$[\hat{M}_{(k)}, D] = -k\hat{M}_{(k)} + (c_{k0} + k\phi_k)1 \tag{193}$$

under the replacement $\hat{M}_{(k)} = M_{(k)} + \phi_k 1$. Therefore, by choosing $\phi_k = -k^{-1}c_{k0}$ we can eliminate $\hat{c}_{k0}$. Then setting $k = 0$ in eq.(190) we find

$$(m+n)c_{mn} = 0 \tag{194}$$

so that the central charge c_{mn} vanishes unless $n = -m$. To find the magnitude of the charge (as in [1], p80) we return to eq.(190) with $c_{mn} = A(m)\delta_{m+n}^0$

$$\delta_{k+m+n}\left[(m-k)A(n) + (n-m)A(k) + (k-n)A(m)\right] = 0 \tag{195}$$

Noting that $A(-m) = -A(m)$, we set $k = 1$, $n = -(m+1)$ to find the recursion relation

$$A(m+1) = \frac{1}{(m-1)}\left[(m+2)A(m) - (2m+1)A(1)\right] \tag{196}$$

The solution,

$$A(m) = am^3 + bm \tag{197}$$

dependent on two initial values, $A(1) \equiv a + b$ and $A(2) \equiv 8a + 2b$, is easily checked by induction.

Finally, we note that a projective representation also allows the introduction of central charges into the commuting Lie algebra $\mathcal{J}$. The consistency of *abitrary* charges with eq.(189) is trivial since all of the structure constants vanish. We can therefore write

$$[J_m, J_n] = b_{mn}1 \tag{198}$$

for any constants $b_{mn} = -b_{nm}$. However, since the right-hand side has vanishing conformal weight, we must have

$$b_{mn} = b_m \delta^0_{m+n} \tag{199}$$

with $b_m = -b_{-m}$. This remaining constant may be absorbed into the definition of the J_m, except for the antisymmetry. The antisymmetry can be accommodated by scaling J_m so that $b_{mn} = m\delta^0_{m+n}$, which gives the usual mode algebra

$$[J_m, J_n] = m\delta^0_{m+n}1 \tag{200}$$

and permits the usual representation of the Virasoro algebra in terms of J_m :

$$L_n = \frac{1}{2} \sum_{n=-\infty}^{\infty} J_{m-n}J_n \tag{201}$$

so the unique fully non-commuting projective Lie algebra is a sub-algebra of the free algebra of the unique fully commuting projective Lie algebra with central extension. This representation of $\mathcal{L}_N$ in terms of $\mathcal{J}$ introduces the usual operator ordering ambiguity for L_0.

It is intriguing to note that the algebra $\mathcal{J}$ with central charges has the form of a canonical quantum commutator, suggesting canonical quantum commutators can be viewed as projective central extensions of commuting classical operators. In fact, this result is shown in [2] for the case of canonical commutators for position and momentum variables in the quantum mechanical limit of the tower structure. The demonstration is comparable for any canonically (conformally!) conjugate pair of classical variables. Also, the parallel transport equation for $(0, 1)$ tensors on a tower geometry is equivalent to the Schrödinger equation (see [2], and below). *It appears that the entire canonical quantization procedure can be formulated in the following way: extend the weak equivalence algebra of classical, commuting operators by strengthening the equivalence relation and considering the projective representations allowed by scale invariance, then let the resulting fields evolve by parallel transport determined by the connection on the full tangent tower.*

7   Conformal dynamics

Before considering vector valued weight operators, i.e., rank $(1, 2)$ tensors, we explore some properties of dynamics on the weight tower. In order to arrive at stringlike structures,

we will need to implement the steps outlined at the end of Sec.(6) for the description of $(1, 2)$ tensor fields on scale-invariant spacetimes. This procedure will be easier if we first consider the dynamics of a simple classical variable, the position of a point particle.

To begin, we need a description by classical commuting variables on the weakly equivalent tower. There is a step implicit in this, because normally, classical fields are described using Weyl equivalence rather than weak equivalence. To see this point clearly, consider the free particle position, x^μ. Even if we regard x^μ as a weight-1 tower scalar, it is not an element of the Hilbert space of the weight tower because it is not self-dual, hence not of positive norm. The simplest way to make it self-dual is to form a new vector by taking the self-dual part,

$$\eta^\mu(x) = \frac{1}{2}(x_{+1}^\mu e^{ix} + x_{-1}^\mu e^{-ix}) \tag{202}$$

Here x_{+1}^μ is the original coordinate, while x_{-1}^μ is its weight-(-1) dual, and we have used the continuous representation. Since weight (-1) means the extra part of the field has units of inverse length, it is trivial to guess that the quantity x_{-1}^μ should be identified with a multiple of $\frac{p^\mu}{\hbar}$. The proportionality constant turns out to require a factor of i because of the difference between the zero signature of the Killing metric of the conformal group and the +4 signature of phase space, but we need not go into this here. A more detailed discussion is given in [2].

The essential point here is that the restriction to self-dual fields required by the conformal structure automatically introduces a combination of position and momentum as the principal variable of the theory. This gives us a deeper understanding of the origin of the uncertainty principle. We now see the coupled evolution of position and momentum as arising from our need to use an inner product to form conformally invariant scalars as candidates for measurement, and from the convergence criterion for vectors needed even to define the weight tower.

We can easily regard η^μ as a vector on either a weakly equivalent or a strongly equivalent weight tower. Choosing strong equivalence as the most predictive (and also the experimentally verified) tower structure, the final step is to look at parallel transport of η^μ on the spacetime manifold, using the tower connection. As shown in [2], parallel transport along a curve with tangent u^μ is equivalent to solving the Lorentz-covariant Schrödinger equation. To see how this arises, consider the form of the covariant exterior derivative of a tower vector. For any gauge covariant derivative,

$$\mathbf{D}\phi = \mathbf{d}\phi + \phi\mathbf{A} \tag{203}$$

where the gauge vector $\mathbf{A}$ is a Lie algebra valued 1-form. For dilations, $\mathbf{A}$ is given by the Weyl vector

$$\theta = \mathbf{d}x^\mu\theta_\mu D \tag{204}$$

where D is the dilation operator. Using our matrix expression

$$(D)^m{}_n = \sum_{s=-\infty}^{\infty} s\delta_s^m \delta_n^s \tag{205}$$

we see that when $\mathbf{D}$ acts on a weight-n vector $\phi_{(n)}$ we get the usual formula for the Weyl-covariant derivative given by eq.(22). The tangent tower formalism combines into a single expression the multiplicity of different Weyl-covariant derivatives represented by eq.(22). Additionally, we can now differentiate indefinite-weight objects such as η^μ:

$$D_\nu \eta^\mu = \partial_\nu \eta^\mu - \eta^\beta \Gamma^\mu_{\beta\nu} + \eta^\mu \theta_\nu D \equiv \mathcal{D}_\nu \eta^\mu + \theta_\nu \eta^\mu D \tag{206}$$

where $\mathcal{D}_\nu$ is the usual covariant derivative using the Christoffel connection, $\Gamma^\mu_{\beta\nu}$. For parallel transport of η^μ we write

$$u^\nu \mathcal{D}_\nu \eta^\mu = i\mathcal{H}\eta^\mu \tag{207}$$

where $i\mathcal{H} = u^\nu \theta_\nu D$ is the component along u^ν of the operator-valued gauge vector and we have used $\eta^\mu D = -D\eta^\mu$ (see eqs.(134,135)). As mentioned above, the factor of i arises in changing from the zero signature of the Killing metric of the conformal group to the signature-4 metric of the usual phase space variables [2]. The identification of $\mathcal{H}$ as the Hamiltonian operator is consistent with the interpretation given in [2] and [3] and also in earlier related, but distinct, work [5]. With this identification, eq.(207) is the Schrödinger equation.

A useful relationship between spacetime and internal parameters is provided by using the continuous representation of the dilation operator in the Schrödinger equation. Consider the action of $D(x, y)$ on a general vector, $f(y)$:

$$Df(x) = \int (-i\partial_x \delta(x - y))f(y) = -i\partial_x f(x) \tag{208}$$

or simply

$$D = -i\frac{\partial}{\partial x} \tag{209}$$

Combining eq.(209) with eq.(207), in flat space so that $\frac{D}{d\tau} = \frac{d}{d\tau}$, we have

$$u^\nu \mathcal{D}_\nu \eta^\mu = \frac{d\eta^\mu(\tau; x)}{d\tau} = i(u^\nu \theta_\nu)\frac{\partial}{\partial x}\eta^\mu(\tau; x) \tag{210}$$

Since $\theta \equiv u^\nu \theta_\nu$ is implicitly a function of τ, eq.(210) provides an explicit map, Φ, unique up to reparameterization, between the parameter, x, of the continuous representation, and the proper time coordinate, τ, along a curve in spacetime. This map provides an embedding of the parameter space $[-\pi, \pi]$ into the spacetime manifold, $\mathcal{M}$, which is necessarily $1 - 1$. As we shall see in the next section, this same embedding also relates $(1, 2)$ tensors to string.

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Canonical commutators for conformally conjugate variables hold if we use the correspondence developed in Appendices 4 and 5 to turn the vector η^μ into a sum of *definite weight* operators. Taking the Φ into account, we begin with the components of the $(1,1)$-vector (η^0, η^k),

$$x^0 \;=\; t = x = i\sum \frac{(-)^m}{m} e^{imx} \tag{211}$$

$$x^k \;=\; x_1^k e^{ix} + x_{-1}^k e^{-ix} \qquad (k = 1, 2, 3) \tag{212}$$

and map to the corresponding definite weight operators

$$T \;=\; tJ_0 = i\sum \frac{(-)^m}{m} J_m(x - y) \equiv \sum T_m \tag{213}$$

$$X^k \;=\; x^k J_0 = \frac{x_1^\mu}{2} J_1 + \frac{x_{-1}^\mu}{2} J_{-1} \tag{214}$$

For the spatial components we can the identify conjugate operators ("position" and "momentum" operators of weight $(+1)$ and (-1), respectively),

$$\eta_1^k \;=\; \frac{1}{2} x_1^k e^{ix} \sum_n e^{in(x-y)} = \frac{x_1^k}{2} J_1 \tag{215}$$

$$\eta_{-1}^k \;=\; \frac{1}{2} x_{-1}^k e^{-ix} \sum_n e^{in(x-y)} = \frac{x_{-1}^k}{2} J_{-1} \tag{216}$$

These operators commute, except for the central term allowed by using a projective representation, in which case we have the usual canonical commutators

$$[\eta_{+1}^k, \eta_{+1}^{k'}] \;=\; [\eta_{-1}^k, \eta_{-1}^{k'}] = 0 \tag{217}$$

$$[\eta_{+1}^k, \eta_{-1}^{k'}] \;=\; \frac{1}{4} x_1^k x_{-1}^{k'} \delta^{kk'} \mathbf{1} \tag{218}$$

For the time coordinate, however, there is no simple single-mode operator. There is instead an infinity of operators T_m satisfying

$$[T_m, T_n] = \frac{1}{m} \delta_{m+n}^0 \mathbf{1} \tag{219}$$

Because the Schrödinger equation gives us a timelike embedding of the parameter space into spacetime, the time operator does not satisfy the same simple commutation with energy that holds for the spatial components with momentum, but instead reproduces the entire $\mathcal{J}$ algebra.

We also note briefly the algebra associated with the self-dual form of $\eta^\mu(x)$. Beginning with the self-dual operator

$$\eta^\mu(x, y) \;\equiv\; \frac{1}{2}\left(\eta^\mu(x - y) + \eta^\mu(-(x + y))\right) \tag{220}$$

$$=\; \frac{1}{2} \sum_{k=-1,1} x_k^\mu \cos kx \; e^{-iky} \tag{221}$$

$$=\; \frac{1}{4}\left(x_{+1}^\mu e^{i(x-y)} + x_{-1}^\mu e^{-i(x-y)} + x_{+1}^\mu e^{-i(x+y)} + x_{-1}^\mu e^{i(x+y)}\right) \tag{222}$$

we extract the operators

$$
\begin{aligned}
J_1 &= e^{i(x-y)} \\
J_{-1} &= e^{-i(x-y)} \\
K_1 &= e^{-i(x+y)} \\
K_{-1} &= e^{i(x+y)}
\end{aligned}
\tag{223}
$$

Since the operators K_1 and K_{-1} are 0-weight K-type operators they satisfy normal rather than t'Hooft commutation relations. Because of this, $J_{1,-1}$, $K_{1,-1}$ form a 4-dim Lie algebra, which can be shown to be $gl(2, R)$ or $u(1) \times su(1, 1)$.

8 String representations of vector valued weight operators

Before venturing into the *uniqueness* of the relationship between moving quantized string and the $(1, 2)$ tensors, we note that the *existence* of such a relationship has already been established. The crucial elements of string theory have already been described as elements of scale-invariant geometry: we have the mode algebra, $\mathcal{J}$, and the Virasoro algebra, both with central charges, acting on Hilbert space. Since the $(1, 2)$ tensors are vector valued weight maps, those maps will be elements of either the mode or the Virasoro algebras. We could simply consider the case when the maps are in the mode algebra, and follow the standard path for describing string states. In this sense, our goal is accomplished. The motion of strings in spacetime could also be fixed by fiat, by setting the Lorentz vector part of the tensor to have the particular properties of a string world sheet. But, as we now show, these properties follow automatically from the map Φ defined in Sec.(7) together with the requirement of measurabity of states. With this goal in mind, we turn to a study of $(1, 2)$ tensors.

Rank $(1, 2)$ tensors have the general form $P^{\mu m}{}_n$ or $P^\mu(x, y)$. There are two reasons (aside from looking like string variables) that the type $(1, 2)$ tensor fields are especially important. First, they may be used to construct arbitrary, higher order tensorial weight maps of rank $(r, 2)$ simply by taking products of maps. Second, they provide a correspondence between weight maps and vector fields. The correspondence permits us to move freely back and forth between tensor fields on spacetime and fields of operators on Hilbert space. We will be particularly interested in the case where this map is $1 - 1$.

Our procedure in this section resembles the Hiesenberg picture of quantum mechanics, in which it is the operators rather than the states which evolve. We assume that these operators can act on general states, with the properties of the operators reflected in the result. Before we study the evolution of these $(1, 2)$ operators, we show how to build convergence into them.

From this Heisenberg point of view, the most general measurable $(1, 2)$ tensor is one which produces an element of the self-dual Hilbert space from a general vector. We therefore ask for the most general $(1, 2)$ tensor field $P^\mu(x, y)$ for which $(P^\mu f)(x)$ is self-dual for all vectors $f(x)$. Such an operator will be called self-dual. Ideally, $P^\mu(x, y)$ would also be required to be of definite weight, but we can easily see from the matrix representation that a self-dual operator must be symmetric about the $m = 0$ row, top to bottom. Such an object cannot be of definite weight, but can be of definite weight plus anti-weight, a property which we will call *semi-definite-weight*. Beginning with the general form for a semi-definite-weight-k operator $P^\mu_{(k)}(x, y)$ and a general vector $f(x)$,

$$P^\mu_{(k)}(x, y) = e^{ikx} \sum_{m=-\infty}^{\infty} \left(\beta^{\mu m} e^{im(x-y)} + \gamma^{\mu m} e^{im(x+y)} \right) \tag{224}$$

$$f(x) = \sum_{m=-\infty}^{\infty} c^m e^{imx} \tag{225}$$

the product is

$$(P^\mu_{(k)} f)(x) = \frac{1}{2\pi} e^{ikx} \int_{-\pi}^{\pi} dy \left(\beta^{\mu m} e^{im(x-y)} + \gamma^{\mu m} e^{im(x+y)} \right) c^k e^{iky} \tag{226}$$

$$= \sum_m e^{ikx} \left(\beta^{\mu m} c^m e^{imx} + \gamma^{\mu m} c^{-m} e^{imx} \right) \tag{227}$$

$$= \sum_n \left(\beta^{\mu,n-k} c^{n-k} + \gamma^{\mu,n-k} c^{-n+k} \right) e^{inx} \tag{228}$$

The product is therefore self-dual provided

$$\left(\beta^{\mu,n-k} c^{n-k} + \gamma^{\mu,n-k} c^{-n+k} \right) = \left(\beta^{\mu,-n-k} c^{-n-k} + \gamma^{\mu,-n-k} c^{n+k} \right) \tag{229}$$

for all c^n. Now, $c^{n-k}, c^{-n+k}, c^{-n-k}$ and c^{n+k} are independent, leaving no solutions, unless either $n = 0$ or $k = 0$. In the first case the expression holds identically, while for $k = 0$ we must have

$$\beta^{\mu n} = \gamma^{\mu,-n} \equiv \frac{1}{2} \alpha^{\mu n} \tag{230}$$

for all $n \neq 0$. The most general self-dual, semi-definite weight operator is therefore either

$$P^\mu_{(0)}(x, y) = \alpha^{\mu 0} + \frac{1}{2} \sum_{m \neq 0} \alpha^{\mu m} \left(e^{im(x-y)} + e^{-im(x+y)} \right) \tag{231}$$

$$= \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \cos(mx) \, e^{-imy} \tag{232}$$

if the semi-definite weight is zero, or

$$F^\mu_{(k)}(x, y) = \lambda^\mu_k e^{iky} + \lambda^\mu_{-k} e^{-iky} = F^\mu_{(k)}(y) \tag{233}$$

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if the operator is of semi-definite weight k. In order to predict measurable quantities, we will be interested in the zero-semi-weight operators.

Self-duality is not the only condition required for positive norm, because $P_{(0)}^{\mu}(x,y)$ is a vector field as well as an operator. The full norm therefore involves the Minkowski inner product as well as the Hilbert inner product. Clearly, requiring $P_{(0)}^{\mu}(x,y)$ to be either spacelike or timelike will keep the sign definite. Of these, only the (forward) timelike vectors are closed under addition. More importantly, we want to parallel transport states in the direction of $P_{(0)}^{\mu}(x,y)$ using eq.(207), which requires that $P_{(0)}^{\mu}(x,y)$ be timelike.

Next we consider the evolution states along $P_{(0)}^{\mu}(x,y)$. For this purpose we require the integral curves, $X^{\mu}(\tau;x,y)$ of $P_{(0)}^{\mu}(x,y)$. While integral curves are well-defined for vector fields, the development of the operator part of $X^{\mu}(\tau;x,y)$ remains to be specified. To this end we require the particular integral operator-curve $X^{\mu}(\tau;x,y)$ such that states $X^{\mu}f$ evolve along P^{μ} by parallel transport. We will call such a set of curves the *integral curves by parallel transport*, or more simply, the *integral transport curves*.

To be integral curves of P^{μ}, the curves must satisfy[10]

$$\frac{d}{d\tau}X_{(0)}^{\mu}(\tau;x,y) = P_{(0)}^{\mu}(x,y) \tag{235}$$

Then, for arbitrary $f(x)$, we require the parallel transport equation, eq.(207) of Sec.(7), to evolve the tower vector $X^{\mu}f$:

$$u^{\nu}\mathcal{D}_{\nu}(X_{(0)}^{\mu}f) + (X_{(0)}^{\mu}f)\theta D = 0 \tag{236}$$

where we have written $\theta = \theta(\tau)$ for $u^{\nu}\theta_{\nu}$. Commuting $fD = -Df$, and letting all spacetime dependence reside in the operator valued coordinate, $X_{(0)}^{\mu} = X_{(0)}^{\mu}(\tau;x,y)$ rather than in $f = f(x)$, the transport equation becomes

$$\frac{1}{\theta(\tau)}\frac{\mathcal{D}X_{(0)}^{\mu}}{d\tau}f = X_{(0)}^{\mu}Df \tag{237}$$

or since f is arbitrary,

$$\frac{1}{\theta(\tau)}\frac{\mathcal{D}X_{(0)}^{\mu}}{d\tau} = X_{(0)}^{\mu}D \tag{238}$$

Finally, applying eq.(135) and reparameterizing the spacetime path, we have

$$\frac{\mathcal{D}X_{(0)}^{\mu}(x,y)}{d\lambda} \equiv \frac{1}{\theta(\tau)}\frac{\mathcal{D}X_{(0)}^{\mu}(x,y)}{d\tau} = i\frac{\partial}{\partial y}X_{(0)}^{\mu}(x,y) \tag{239}$$

---

[10]The proof that X^{μ} has semi-weight zero is simply to write is as an explicit sum of terms in $(x-y)$ or $(x+y)$,

$$X_{(0)}^{\mu}(x,y) = x^{\mu} + \frac{1}{2}\alpha_{0}^{\mu}\left((x+y) - (x-y)\right) + i\sum\frac{1}{m}\alpha_{m}^{\mu}e^{im(x-y)} + i\sum\frac{1}{m}\alpha_{m}^{\mu}e^{-im(x+y)} \tag{234}$$

Each of these terms either commutes or anticommutes with D.

In flat spacetime, with $\frac{\mathcal{D}}{d\lambda} \to \frac{d}{d\lambda}$, we can identify

$$\frac{d}{d\lambda} = \frac{1}{\theta(\tau)}\frac{d}{d\tau} = i\frac{\partial}{\partial y} \tag{240}$$

or

$$y = i\int \theta(\tau)d\tau = i\lambda \tag{241}$$

that is, we embed the y-parameter into $\mathcal{M}$ as before. Succinctly put, $X^\mu(\tau; x_0, y)$ is the lifting of the integral curves $X^\mu(\tau)$ of $P^\mu(\tau; x_0, y)$ into the tower bundle in such a way that $y = i\lambda$. The factor of i in these expressions is again due to the use of the physical variable $u^\mu = \frac{dx^\mu}{d\lambda}$ instead of the geometric variable iu^μ which arises from conformal gauge theory. In the geometric picture, this is a real-valued embedding [3] and the factor i is part of the signature of the space.

Now suppose that in some region, $\mathcal{S}$, of flat spacetime (so that $\frac{\mathcal{D}}{d\tau} \to \frac{d}{d\tau}$) we have a non-constant, Lorentz-timelike, zero-weight C^∞ operator field $P^\mu_{(0)}(x^\alpha; x, y)$. Fix $x = x_0$, let an initial point $\mathcal{P}$ have spacetime coordinates x^μ, and suppose $X^\mu_{(0)}(\lambda; x_0, y)$ is the integral transport curve of $P^\mu_{(0)}$, that is,

$$\frac{dX^\mu_{(0)}}{d\lambda} = i\frac{\partial X^\mu_{(0)}}{\partial y} = P^\mu_{(0)} \tag{242}$$

Eq.(242) is easily integrated using the explicit form

$$P^\mu_{(0)}(x^\alpha; x, y) = i\sum \alpha^{\mu m}(x^\alpha)\cos mx\ e^{-imy} \tag{243}$$

for $P^\mu_{(0)}$, giving the integral transport curve,

$$X^\mu_{(0)}(x_0, y = i\lambda) = x^\mu(\mathcal{P}) + \alpha^{\mu 0}y + i\sum_{m\neq 0}\frac{1}{m}\alpha^{\mu m}\cos(mx_0)\ e^{-imy} \tag{244}$$

which is precisely eq.(2) for the open string evaluated along a timelike curve, $\sigma = x_0$. The integration produces a self-dual vector field. Notice that the linear function of y (or the polynomial that arises from further integration) is consistent with self-duality because any function of y alone is a self-dual operator (see also eq.(233)). A function of x alone is not self-dual.

When we perform the integral above for each point x in $\mathcal{S}$, we get a series of curves $X^\mu_{(0)}(x, y)$ with tangent vectors $P^\mu_{(0)}(x, y)$. This gives an injective map Φ^μ defined by

$$\Phi^\mu \quad : \quad \mathcal{S} \to \mathcal{M} \tag{245}$$
$$\Phi^\mu(x, y) \quad \equiv \quad X^\mu(x, y) \tag{246}$$

The injected set has dimension ≤ 2. We now prove that Φ^μ is an immersion of the entire region $\mathcal{N}_\pi \equiv (0, \pi) \times (0, \pi) \subset \mathcal{N}$ into $\mathcal{M}$ (and therefore of dimension 2).

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At a general value of the pair (x, y) consider the tangent space to the injected set $X_{(0)}^\mu(x, y)$. This space $\mathcal{V}_2$ consists of all vectors

$$V^\mu(\alpha, \beta) = \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}\right) X_{(0)}^\mu(x, y) \tag{247}$$

$$= \alpha \left(-i \sum_{m \neq 0} \alpha^{\mu m} \sin(mx) \ e^{-imy}\right) + \beta \left(\alpha_0^\mu + \sum_{m \neq 0} \alpha^{\mu m} \cos(mx) \ e^{-imy}\right) \tag{248}$$

which will be 2-dimensional provided the basis vectors

$$Q^\mu = \left(-i \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \sin(mx) \ e^{-imy}\right) \tag{249}$$

$$P^\mu = \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \cos(mx) \ e^{-imy} \tag{250}$$

are independent.

We begin by looking at the inner product $P^\mu Q_\mu$:

$$P^\mu Q_\mu = -i \sum_{m,n=-\infty}^{\infty} \alpha^{\mu m} \alpha \alpha_\mu^n \cos(mx) \ \sin(nx) \ e^{-i(m+n)y} \tag{251}$$

The action of $P^\mu Q_\mu$ on an arbitrary self-dual vector $f = \sum c^k e^{ikx}$ is given by

$$P^\mu Q_\mu f = -\frac{i}{2\pi} \int dy \sum_{k,m,n=-\infty}^{\infty} \alpha^{\mu m} \alpha_{\mu n} \cos(mx) \ \sin(nx) \ e^{-i(m+n-k)y} c^k \tag{252}$$

$$= -i \sum_{m,n=-\infty}^{\infty} \alpha^{\mu m} \alpha_\mu^n c^{m+n} \cos(mx) \ \sin(nx) \tag{253}$$

$$= -i \sum_{m,n=-\infty}^{\infty} \alpha_m^\mu \alpha_{\mu n} c^{-m-n} \cos(-mx) \ \sin(-nx) \tag{254}$$

$$= +i \sum_{m,n=-\infty}^{\infty} \alpha^{\mu m} \alpha_\mu^n c^{m+n} \cos(mx) \ \sin(nx) \tag{255}$$

$$= 0 \tag{256}$$

where we use the self-dual representation $\alpha^{\mu m} = \alpha_m^\mu$ and $c^k = c_k$ in the penultimate step (see Appendix 4). Therefore, $P^\mu Q_\mu = 0$ on all measurable vectors, and so is equivalent to the zero operator. Now, setting eq.(251) for $P^\mu Q_\mu$ to zero explicitly, we can multiply both sides by e^{iky} and integrate over y to find

$$0 = -i \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \alpha_\mu^{k-m} \cos(mx) \ \sin(k-m)x \tag{257}$$

$$= -\frac{i}{2} \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \alpha_\mu^{k-m} \left(\sin kx - \sin(2m-k)x\right) \tag{258}$$

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Now, multiplying by $\sin nx$ and integrating over x gives two nonzero cases. When $n = k$, we have[11]

$$\sum_{m=-\infty}^{\infty} \alpha^{\mu m} \alpha_{\mu}^{k-m} = 0 \tag{259}$$

while $n = 2m - k$ gives an equivalent result. Working backward, we write

$$0 = \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \alpha_{\mu}^{k-m} \cos kx \tag{260}$$

$$= \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \alpha_{\mu}^{n} \cos kx \ \delta_{k-m}^{n} \tag{261}$$

$$= \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \alpha_{\mu}^{n} \cos(m+n)x \ \frac{1}{2\pi} \int e^{-i(n+m)z} e^{ikz} dz \tag{262}$$

This expression, multiplied by e^{-iky}, then summed on k, gives a δ-function, $2\pi\delta(y-z)$. Performing the z-integration then leads to

$$0 = \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \alpha_{\mu}^{n} \cos(m+n)x \ e^{-i(n+m)y} \tag{263}$$

This same expression arises if we compute $P^2 + Q^2$,

$$P^2 + Q^2 = \sum_{m,n=-\infty}^{\infty} \alpha^{\mu m} \alpha_{\mu}^{n} (\cos(mx) \ \cos(nx) \ - \sin(mx) \ \sin(nx)) \ e^{-i(m+n)y} \tag{264}$$

$$= \sum_{m=-\infty}^{\infty} \alpha^{\mu m} \alpha_{\mu}^{k-m} \cos(m+n)x \ e^{-i(n+m)y} \tag{265}$$

$$= 0 \tag{266}$$

so that $P^{\mu}P_{\mu} = -Q^{\mu}Q_{\mu} < 0$, where the inequality holds because P^{μ} is our original timelike $(1,2)$ tensor field. Therefore, Q^{μ} is a nonvanishing spacelike $(1,2)$ vector field orthogonal to P^{μ}, and the immersion is established at every point of $(0,\pi) \times (-\pi,\pi)$. At the endpoints $x = 0$ or π, Q^{μ} vanishes because $\sin n\pi$ vanishes. The map Φ^{μ} is therefore an immersion of the region $\mathcal{N}_{\pi}$ into $\mathcal{M}$, where it is convenient to restrict y to the same interval as x, setting $\mathcal{N}_{\pi} = (0,\pi) \times (0,\pi)$.

As a consequence of the immersion Φ^{μ}, the Lorentzian metric structure of $\mathcal{M}$ induces a Lorentzian structure on $\mathcal{N}_{\pi}$. To establish the Lorentzian structure on $\mathcal{N}_{\pi}$, we start with vectors on $\mathcal{N}_{\pi}$, using the parameters (x,y) as a basis. Letting

$$\sigma^{a} \equiv (x, y) \tag{267}$$

we have the general vector

$$v = v^{a} \frac{\partial}{\partial \sigma^{a}} \tag{268}$$

---

[11]Once the $P^{\mu m}$ are represented with central charges, the $\alpha^{\mu m}$ acquire true operator status. Then eq.(259) corresponds to the physical state conditions $L_{m}\phi = 0$ for string.

The map $\Phi^\mu = X^\mu(x,y)$ pushes this vector forward to a vector V tangent to the immersed 2-surface in $\mathcal{M}$,

$$V = v^a \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial}{\partial x^\mu} = V^\mu \frac{\partial}{\partial x^\mu} \tag{269}$$

The induced metric on the immersed 2-surface is now pulled back from $\mathcal{M}$ by defining $v^a v_a$ to be equal to $V^\mu V_\mu$,

$$V^\mu V_\mu = g_{\mu\nu} V^\mu V^\nu = g_{\mu\nu} \left(v^a \frac{\partial X^\mu}{\partial \sigma^a} \right) \left(v^b \frac{\partial X^\nu}{\partial \sigma^b} \right) \tag{270}$$

$$\equiv h_{ab} v^a v^b \tag{271}$$

so h_{ab} is given by

$$h_{ab} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} = \begin{pmatrix} P^2 & P^\mu Q_\mu \\ P^\mu Q_\mu & Q^2 \end{pmatrix} = Q^2 \eta_{ab} \tag{272}$$

Since the tangent space to $\Phi^\mu(\mathcal{N}_\pi)$ is just $\mathcal{N}_\pi$, h_{ab} provides a metric on both the immersed 2 surface and on $\mathcal{N}_\pi$. We can give this expression manifest 2-dim conformal invariance by using the inverse metric, h^{ab}, to write

$$2 = h^{ab} h_{ab} = h^{ab} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X_\mu}{\partial \sigma^b} \tag{273}$$

and therefore

$$h_{ab} = \frac{2 \partial_a X^\mu \partial_b X_\mu}{h^{cd} \partial_c X^\nu \partial_d X_\nu} \tag{274}$$

Returning to $\mathcal{N}_\pi$, we next look at the null vectors,

$$h(v,v) = Q^2 \eta_{ab} v^a v^b = 0 \tag{275}$$

Clearly, the two null directions are

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \tag{276}$$

Thus the induced Lorentz metric makes rigorous the "left-moving" and "right-moving" nomenclature for the $(x+y)$ and $(x-y)$ Fourier modes. In terms of h_{ab}, these null modes are left- and right- moving waves, and $P^\mu_{(0)}(x,y)$, $Q^\mu(x,y)$ and $X^\mu_{(0)}(x,y)$ all satisfy the wave equation,

$$\Box_h X^\mu_{(0)}(x,y) \equiv \frac{1}{\sqrt{-h}} \partial_a \left(\sqrt{-h} h^{ab} \partial_b X^\mu_{(0)}(x,y) \right) = 0 \tag{277}$$

Similarly, it is a simple exercise to check directly that a Lorentz transformation Λ on $\mathcal{M}$ induces a corresponding Lorentz transformation on $\mathcal{N}_\pi$.

The action of Φ^μ on $\mathcal{N}_\pi$ gives the world sheet

$$X^\mu_{(0)}(x,y) = x^\mu(\mathcal{P}) + \alpha^\mu_0 y + i \sum_{m \neq 0} \frac{1}{m} \alpha^{\mu m} \cos(mx) \, e^{-imy} \tag{278}$$

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Eq.(278) will be called the *string representation* of the zero-semi-weight self-dual $(1,2)$ tensor. While eq.(278) has the appearance of a standard Fourier series, it differs in an important respect, namely, that the parameters σ^a are labels for the space of operators on the weight tower. The explicit m-dependence of the coefficient $\alpha^{\mu m}$ determines which algebra the operator $\alpha^{\mu m} \cos(mx) \, e^{-imy}$ belongs to. By contrast, the amplitude, $\alpha^{\mu m}$ *is* the entire operator in the usual quantized string expansion, eq.(2). However, this picture changes when we look at the details of the mode algebra.

To see the emergence of the usual mode algebra, i.e., the canonical commutators of the mode amplitudes, we follow the example of the single particle above, mapping to a set of definite weight operators. Following the steps of Appendix 5 we insure a unique mapping by starting with the self-dual operator $P_0^\mu(x,y) = v^\mu(x-y) + v^\mu(x+y)$, extracting the $(1,1)$ vector $v^\mu(x) = \sum \alpha^{\mu m} e^{imx}$, then mapping from $v^\mu(x)$ to the associated definite weight series,

$$V^\mu \equiv v^\mu(x) J_0(x,y) = \sum_m \alpha^{\mu m} \, J_m = \sum_m V_{(m)}^\mu \tag{279}$$

These operators therefore commute except for possible central extensions. Using the self-dual form with the maximal central extension as developed in Appendix 5, the commutation algebra is the usual string mode algebra,

$$[V_{(m)}^\mu, V_{(n)}^\nu] = m(V \cdot V)\delta_{m+n}^0 \eta^{\mu\nu}\mathbf{1} \tag{280}$$

provided only that we normalize the amplitudes $(V \cdot V)$ to unity.

At this point, the question arises of how to represent the mode algebra with central charges. The Fourier exponentials, while operators, all commute and therefore may be treated as regular numbers. Consequently, there is no harm in treating the $\alpha^{\mu m}$ as the projective representation of the algebra, since a different representation is required anyway. There remains the difference from standard string theory, that here the mode algebra, like any set of canonical commutators, is seen only in the associated definite-weight algebra. Nonetheless, the associated algebra may be used in the standard way to specify physical states, construct the Virasoro and Poincaré generators and so on. Moreover, as we summarize in our final theorem below, both expressions describe extremal world sheets moving in spacetime.

Before moving to the theorem, we note that the operators present in eq.(278) for the string representation of $X^\mu(x,y)$ also form the self-dual algebra associated with the mode algebra, in a way perfectly parallel to the $gl(2,R)$ algebra associated with the self-dual commutators of Sec.(6), eqs.(223). This time the algebra, with basis

$$e^{im(x-y)}, e^{im(x+y)} \tag{281}$$

obviously generates the analytic functions on $\mathcal{N}$. For each pair $(m,-m)$ the algebra contains a copy of $gl(2,R)$. The restriction of this algebra to the combination of generators

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which actually occurs in X^μ, namely

$$S_m = \cos mx \; e^{-imy} \tag{282}$$

generates those functions on $\mathcal{N}$ which have cosine expansions for the x variable.

We conclude by showing that the $(1,2)$ tensor field X^μ satisfies the X^μ, h_{ab}, and endpoint variations of the string action

$$S = \int \sqrt{h} h^{ab} g_{\mu\nu} \partial_a \alpha^\mu \partial_b \alpha^\nu \tag{283}$$

This means that like string theory, the conformal dynamics of $(1,2)$ tensors describes an extremal world sheet moving in spacetime.

Theorem 4 *4: The integral transport curves of any timelike, zero-weight, C^∞, self-dual operator field $P^\mu_{(0)}(x^\alpha; x, y)$ provide an immersion of $\mathcal{N}_\pi$ into $\mathcal{M}$, causal in the pull-back of the spacetime metric, such that the immersed 2-surface extremizes the action*

$$S = \int \sqrt{h} h^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{284}$$

with respect to δh^{ab} and δX^μ, including the boundary condition.

Proof The variation δX^μ leads to

$$\Box_h X^\mu_{(0)}(x, y) = 0, \tag{285}$$

where the surface term vanishes provided

$$Q^\mu(0) = Q^\mu(\pi) = 0, \tag{286}$$

and the δh^{ab} variation leads to

$$\partial_a X^\mu \partial_b X_\mu = \frac{1}{2} h_{ab} (h^{cd} \partial_c X^\mu \partial_d X_\mu) \tag{287}$$

All of these results have been shown above to follow from the conformal dynamics of the integral transport curves of a self-dual $(1,2)$ tensor $P^\mu_{(0)}(x, y)$ (see eqs.(277), (249) and (274), respectively). The mapping $\Phi^\mu(x, y)$ is necessarily causal because the Lorentz structure on $\mathcal{M}$ is inherited by $\mathcal{N}_\pi$, and $\Phi^\mu(x, y)$ is an immersion because P^μ was assumed to be C^∞ and P^μ and Q^μ are independent throughout $\mathcal{N}_\pi$.

Loosely stated, the theorem tells us that any measurable $(1,2)$ tensor field corresponds to a string solution. Notice the important role played by measurability in the proof. It was the use of self-dual equivalence classes of operators that allowed us to show that $P^\mu Q_\mu = 0$, from which the immersion followed. The metric structure on $\mathcal{N}_\pi$ follows from the immersion. Finally, recall that the importance of using self-dual equivalence classes is that self-duality provides the positive definiteness needed to insure convergence of vectors on the tangent tower. In the final summary section, we review this series of results.

9 Summary

The conformal dynamics of the weight tower associated with any scale-invariant geometry follows directly from a careful treatment of the classical physical principles of scale invariance. The mathematical structures that result from this treatment include a Hilbert space associated with each point of spacetime, as well as two of the most important operator algebras of modern quantum field theory. While there remain many implications of the new formalism to explore, and there are sometimes puzzling differences of representation, the direct emergence of such fundamental elements of physical theory is an important finding. In this final section, we provide a short summary of our major results.

We show how the freedom to choose units for spacetime fields, and our usual assumptions about dimensionful fields, leads to a tangent tower structure associated with any scale-invariant spacetime. Because this tangent tower is an infinite dimensional vector space, we require some convergence criterion in order to define it and in order to form finite, measurable scalars. The criterion we use is convergence in a positive definite norm, which produces a Hilbert space on the tower. In order to guarantee positive definiteness of the norm, only conformally self-dual vectors are allowed.

We next consider mappings on the tangent tower. Since only conformal and Lorentz scalars are measurable, it is necessary to restrict linear mappings on the tangent tower to those of definite or semi-definite conformal weight. Then, the maximally commuting complete Lie algebra of definite-weight operators is the set of unit, shifted-diagonal matrices, while the maximally non-commuting complete Lie algebra of definite-weight operators is the Virasoro algebra.

We develop a continuous representation for weight maps and show that the maps having either vanishing commutators or vanishing anticommutors with the dilation generator, D, together span the space of harmonic functions in 2-dim.

A study of the equivalence relations respecting the tangent tower structure showed the existence of two equivalence relations which can be used to define distinct tower structures. The first, weak equivalence, is consistent only with the fully commuting (up to central charges) Lie algebra of zero conformal weight operators. The operators of this algebra will all be simultaneously measurable. The second equivalence relation, called strong equivalence, allows non-commuting operators of various conformal weights. Since only zero weight objects are measurable, objects on a strongly equivalent tangent tower cannot be simultaneously measured.

We continue with the study of strong equivalence, showing that projective representations of both the mode and Virasoro algebras identified earlier occur with central charges. The projective representations are permitted by both the weak and strong equivalence relations.

Moving to a simple particle example to study conformal dynamics, we show that the parallel transport of an $(r, 1)$ tensor satisfies the Schrödinger equation. The operator

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character of the Schrödinger equation follows because the spacetime connection acts on the entire tangent tower. The factor of i in the Schrödinger equation arises because the natural geometric variables (with 0-signature 8-dim metric) differ by a factor of i from the usual physical position and momentum variables (with two +2-signature 4-dim metrics). The model involves the identification of the Hamiltonian operator with the time component of the dilational gauge vector times the dilation operator. This interpretation has been consistently implemented elsewhere [2],[3],[5].

Paralleling the conformal dynamics of a particle, we find the integral curves of a general timelike, self-dual $(1, 2)$ tensor. To integrate the operator part of the field we demand that the field parallel transports the vectors on which it acts. The resulting mapping from the 2-parameter space $\mathcal{N}_\pi = (0, \pi) \times (0, \pi)$ into spacetime is shown to be an immersion of the same form as the open string solution. This immersion induces a Lorentzian metric on $\mathcal{N}_\pi$ and satisfies the variational equations governing open string.

In addition, a series of appendices develops a range of properties of the new conformal formalism, including alternative representations for tower tensors, the quantum algebra of weight operators, the use of length scales, and many results on general and self-dual operators.

Taken as a whole, our results show that scale invariance provides an underlying, principled reason for the physical importance of Hilbert space, the Virasoro algebra, the string mode expansion, canonical commutators and Schrödinger evolution of states, all of which is independent of the insights of both string theory and quantum theory.

Appendix 1: Alternate bases   The simple form of eq.(116) for $D(x,y)$ leads to a set of definite weight functions $f_k(x)$ with only pure modes $f_k(x) \sim e^{ikx}$, and definite weight operators $M_{(k)}(x,y)$ which for $k \neq 0$ have only "right moving modes", $M_{(k)}(x,y) \sim M_{(k)}(x-y)$. Such heterosis [1] appears to be a necessary concomitant of definite weight fields. To emphasize this fact, we repeat in this Appendix the calculation of Sec.(5), starting with a more general ansatz for the form of $D(x,y)$. As expected, we find that while most choices of basis lead to both right- and left-moving modes, there is always a fixed relationship between the modes so that there remains the same number of degreees of freedom.

We begin by choosing the form of $D(x,y)$ to be an arbitrary superposition of both left- and right-moving Fourier modes, plus a possible correction for surface terms

$$D(x,y) = -i\frac{\partial}{\partial x}(a\delta(x-y) + b\delta(x+y)) - ic(x,y) \tag{288}$$

where

$$c(x,y) = c_1\delta(x+\pi)\delta(y+\pi) + c_2\delta(x+\pi)\delta(y-\pi) \tag{289}$$
$$+c_3\delta(x-\pi)\delta(y+\pi) + c_4\delta(x-\pi)\delta(y-\pi) \tag{290}$$

We seek a family of definite weight bases $f_k(x)$ for a range of values of (a,b). Despite containing both left and right modes, we find that this set of bases still displays heterosis through a coupling of the amplitudes of the two different sets of modes.

Consider the action of $D(x,y)$ on a $(0,1)$ vector

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \exp ikx \tag{291}$$

where, as before, $f(\pi) = f(-\pi)$. Computing Df without $c(x,y)$ we find a surface term of the form

$$\delta(x-\pi)[iaf(\pi) + ibf(-\pi)] - \delta(x+\pi)[ibf(\pi) + iaf(-\pi)] \tag{292}$$

which vanishes only if $a = {}^+_-b$ and $f(\pi) = \mp f(-\pi)$. The condition is even more stingent if we also allow D to act on f from the right, fD . For both surface terms to vanish requires $f({}^+_-\pi) = 0$. In combination with the eigenvalue relation, eq.(118), $f({}^+_-\pi) = 0$ is inconsistent.

Therefore, the singular correction term is again necessary in the definition of D. To find $c(x,y)$, consider the right and left action of D on the rth slot of a rank $(0,k)$ tensor,

$$(DT)_r \equiv \int_{-\pi}^{\pi} dz D(x,z) T(x_1, x_2, \ldots x_{r-1}, z, x_{r-2}, , \ldots x_k) \tag{293}$$

and a similar expression for $(TD)_r$. For $(DT)_r$ the integration by parts gives a surface term of the form

$$
\begin{aligned}
S.T. \quad = \quad & -i\left[(-a\delta(x-z)+b\delta(x+z))T(x_1,x_2,\ldots z\ldots x_k)\right]_{-\pi}^{\pi} \\
& -i\int_{-\pi}^{\pi} c(x,z)T(x_1,x_2,\ldots z\ldots x_k)
\end{aligned}
\tag{294}
$$

which cancels provided

$$
\begin{aligned}
c_1 + c_2 &= a + b \tag{295} \\
c_3 + c_4 &= -a - b \tag{296}
\end{aligned}
$$

Similarly, the surface term in evaluating $(TD)_r$ vanishes provided

$$
\begin{aligned}
c_1 + c_3 &= b - a \tag{297} \\
c_2 + c_4 &= a - b \tag{298}
\end{aligned}
$$

Both conditions are satisfied if D is of the form

$$
\begin{aligned}
D(x,y) \quad = \quad & -ia[\partial_x\delta(x-y)-\delta(x-\pi)\delta(y+\pi)+i\delta(x+\pi)\delta(y-\pi)] \tag{299} \\
& -ib[\partial_x\delta(x+y)-\delta(x-\pi)\delta(y+\pi) \tag{300} \\
& \qquad -\delta(x+\pi)\delta(y-\pi)+2\delta(x-\pi)\delta(y-\pi)] \tag{301}
\end{aligned}
$$

plus an arbitrary multiple of the purely surface term

$$
\begin{aligned}
E \quad = \quad & [\delta(x-\pi)\delta(y-\pi)-\delta(x-\pi)\delta(y+\pi) \tag{302} \\
& -\delta(x+\pi)\delta(y-\pi)+\delta(x+\pi)\delta(y+\pi)] \tag{303}
\end{aligned}
$$

Since $ET = TE = 0$ for any tensor T (as long as $T(\pi) = T(-\pi)$) we can simply drop the E term.

With eq.(301) for D, the left and right action of $D(x,y)$ on the rth slot of any tensor is simply given by the integrated part

$$
\begin{aligned}
(DT)_r &= -ia\frac{\partial T}{\partial z}\mid_{z=x_r} +ib\frac{\partial T}{\partial z}\mid_{z=-x_r} \tag{304} \\
(TD)_r &= ia\frac{\partial T}{\partial z}\mid_{z=x_r} +ib\frac{\partial T}{\partial z}\mid_{z=-x_r} \tag{305}
\end{aligned}
$$

Using eqs.(304) and (305) we now find a complete set of eigenfuntions and definite weight operators.

Substituting $D(x,y)$ into

$$
Df^{(k)} = kf^{(k)} \tag{306}
$$

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we find the differential equation

$$Df^{(k)} = -ia\frac{\partial f^{(k)}(z)}{\partial z} \mid_{z=x_r} +ib\frac{\partial f^{(k)}(z)}{\partial z} \mid_{z=-x_r} = kf^{(k)}(x) \tag{307}$$

which is solved by the set of functions

$$f^{(k)}(x) = A_k(e^{ikx}\cosh\beta/2 - e^{-ikx}\sinh\beta/2) \tag{308}$$

provided a and b are related by

$$a = \cosh\beta \tag{309}$$
$$b = \sinh\beta \tag{310}$$

Fixing the overall normalization of $D(x,y)$ provides us with a 1-parameter family of representations for the dilation operator. Of course, the $\beta = 0$ case reduces D to the form of eq.(116) considered previously. In the special case where $a = b$ the zero mode solution may be any sine series

$$f^{(0)}(x) = \sum_{n=1}^{\infty} A_n \sin nx \tag{311}$$

while for $a = b$, $k \neq 0$ the only solutions are symmetric functions with discontinuous derivatives at the origin.

For $(r,2)$ tensors we can compute the commutator,

$$[D,M](x,y) = (DM)_1 - (MD)_2 \tag{312}$$

which allows us to define operators $M_{(k)}$ of definite weight by demanding

$$[D, M_{(k)}] = kM_{(k)} \tag{313}$$

Substituting from eqs.(304) and (305),

$$a\partial_x M(x,y) - b\partial_z M(z,y) \mid_{z=-x} +a\partial_y M(x,y) + b\partial_z M(x,z) \mid_{z=-y} = ikM(x,y) \tag{314}$$

Separation of variables, $M(x,y) = f(x)g(y)$ leads to a pair of equations of the form of eq.(307), giving the immediate solution:

$$\begin{aligned}
M_{(k)}(x,y) &= \sum_{m=-\infty}^{\infty} \alpha_m^k f^{(k-m)}(x)f^{(m)}(y) \tag{315}\\
&= e^{ikx}h_k(x-y) \tag{316}\\
&\quad + \sinh\beta/2[e^{ikx}(h_k(x-y)\sinh\beta/2 - h_k(x+y)\cosh\beta/2) + c.c.] \tag{317}
\end{aligned}$$

where

$$h_k(z) \equiv \sum_{m=-\infty}^{\infty} \alpha_m^k e^{-imz} \tag{318}$$

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Thus, while $M_{(k)}(x,y)$ in general has both right and left modes, the coefficients are not independent. Instead, they depend on the single function $h_k(z)$ with z replaced by either $x+y$ or $x-y$.

For $\beta = 0$, so that $D(x,y)$ contains only $\delta(x-y)$, the commutator reduces to the previous expression,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) M(x,y) = ikM(x,y) \tag{319}$$

with solution

$$M_{(k)}(x,y) = \sum_{m=-\infty}^{\infty} \alpha_m^k e^{ikx} e^{-im(x-y)} \tag{320}$$

as expected.

Appendix 2: A t'Hooft-Virasoro quantum algebra of weighted operators The operators $M_{(k)}, N_{(k)}, J_{(k)}$ and $K_{(k)}$ defined in Secs.(2) and (3) together form a quantum algebra. Writing the operators in the form

$$M_{(k)}^{1,0} = M_{(k)} = e^{ikx} \sum \alpha_m^k e^{im(x-y)} \tag{321}$$

$$M_{(k)}^{1,1} = N_{(k)} = e^{ikx} \sum \alpha_m^k e^{im(x+y)} \tag{322}$$

$$M_{(k)}^{0,0} = J_{(k)} = e^{ikx} \sum e^{im(x-y)} \tag{323}$$

$$M_{(k)}^{0,1} = K_{(k)} = e^{ikx} \sum e^{im(x+y)} \tag{324}$$

where $\alpha_m^k = m$. The graded t'Hooft commutatorsare defined by

$$[M_{(m)}^{\alpha,\beta}, M_{(n)}^{\alpha',\beta'}] \equiv M_{(m)}^{\alpha,\beta} M_{(n)}^{\alpha',\beta'} - (-)^{\alpha\beta'+\beta\alpha'} e^{2i(\beta'm-\beta n)x} M_{(n)}^{\alpha',\beta'} M_{(m)}^{\alpha,\beta} \tag{325}$$

where $m \geq n$ and $\alpha, \beta, \alpha', \beta' \in \{0, 1\}$. The algebra is then given by

$$[M_{(m)}, M_{(n)}] = (n-m)M_{(m+n)} \tag{326}$$

$$[M_{(m)}, J_{(n)}] = nJ_{(m+n)} \tag{327}$$

$$[M_{(m)}, N_{(n)}]_{tH}^+ = (m+n)N_{(m+n)} \tag{328}$$

$$[M_{(m)}, K_{(n)}]_{tH}^+ = nK_{(m+n)} \tag{329}$$

$$[J_{(m)}, N_{(n)}]_{tH} = mK_{(m+n)} \tag{330}$$

$$[J_{(m)}, J_{(n)}] = 0 \tag{331}$$

$$[J_{(m)}, K_{(n)}]_{tH} = 0 \tag{332}$$

$$[N_{(m)}, N_{(n)}]_{tH} = (n-m)M_{(m-n)} \tag{333}$$

$$[N_{(m)}, K_{(n)}]_{tH}^+ = -nJ_{(m-n)} \tag{334}$$

$$[K_{(m)}, K_{(n)}]_{tH} = 0 \tag{335}$$

where a subscript tH indicates the presence of a non-zero phase, $e^{2i(\beta'm-\beta n)x}$, and a superscript $+$ denotes an anticommutator due to the factor $(-)^{\alpha\beta'+\beta\alpha'}$. Notice that $(M_{(k)}, J_{(k)})$ form a Lie algebra, as do $(J_{(0)}, K_{(0)}, M_{(0)}, N_{(0)})$.

Appendix 3: Length scales and phases   A $(0, 1)$ vector in the continuous representation is given by a Fourier series

$$v(x) = \sum_{m=-\infty}^{\infty} v^k e^{ikx} \tag{336}$$

Whereas the usual definite-weight quantities of physical theory have only a single term, these vectors are superposition of all possible weights. The question then arises, how do we make the connection with our usual units? In this Appendix we address this question.

Measurements are made relative to some standard of length such as a meter stick or a clock. Given a set of meter sticks (of length l) distributed conveniently throughout spacetime, we construct the weight vector $l_{(1)} = l_{(1)}(x^\mu)$. In the continuous representation $l_{(1)}$ becomes $l_1(x^\mu; x) = l_{(1)} e^{ix}$. This field may be used to produce a measurable length from a general tower vector $v(x)$ by taking the inner product. This yields the dimensionless ratio

$$\langle v, l_{(1)} \rangle = \sum v^n l_n = \frac{v^1}{l} \tag{337}$$

which is simply the magnitude of v^1 in units of l.

A related $(0, 1)$ field,

$$\mathcal{L}(x) = \sum (l)^n e^{inx} \tag{338}$$

in which the components l^n are simply powers of l, can be used to interchange all of the phases for the usual units in a general vector $v(x)$. The inner product is the dimensionless expression

$$\langle v, l \rangle = \sum v^n l_n = \sum \frac{v^n}{(l)^n} \tag{339}$$

in which the $(l)^n$-dimensioned component v^n of v is given in units of $(l)^n$.

It is interesting to speculate about the consequences of conformal evolution of $l_1(x)$. Presumably, $l(x^\mu; x)$ will evolve by parallel transport like any other weight vector. But we have no absolute knowledge of the function $l_1(x^\mu; x)$, rather, $l_1(x^\mu; x)$ is our *definition* of unit weight. Thus, even if l_1 evolved into a more general superposition

$$l(x) = \sum l^n e^{inx} \tag{340}$$

any measurement would treat it as a single mode

$$L(x) e^{iy(x)} \equiv \sum l^n e^{inx} \tag{341}$$

effectively defining a new continuous parameter $y = y(x)$. Such repeated redefinitions of the basis may account for "collapse of the wave function."

Appendix 4: Vectors and their associated operators; properties of self-dual operators   There are certain $1-1$ maps which hold between vectors, definite weight operators, and self-dual operators. These maps, and their generalizations to $(r, 1)$ tensors, play an important role in the relationship of conformal dynamics to quantum dynamics. After studying these maps, we look in particular detail at self-dual operators.

The first map, Ψ, relates any any $(0, 1)$ vector, $v(x)$, to a corresponding $(0, 2)$ operator. We can use the identity operator

$$J_0 = \sum e^{im(x-y)} = 2\pi\delta(x - y) \tag{342}$$

multiplicatively to produce the associated operator

$$V(x, y) = v(x)J_0 \tag{343}$$

Writing the expansion $v(x) = \sum \alpha^m e^{imx}$ we see that producing $V(x, y)$ amounts to the replacement of the weight-m vector e^{imx} by the weight-m commuting operator J_m since

$$V(x, y) = v(x)J_0 = \sum_m (\alpha^m e^{imx}) \sum_n e^{in(x-y)} = \sum_m \alpha^m J_m \tag{344}$$

Starting with classical spacetime variables, it is easy to write down corresponding tower vectors. Ψ is then used to relate these vectors in the self-dual Hilbert space to classical, commuting operators, which in turn become non-commuting when the algebra acquires the central extension allowed by projective representations.

A second pair of correspondences produces a 0-weight operator and a self-dual operator simply by varying the argument of $v(x)$. Unlike the previous map, these correspondences do not preserve the conformal weight associated with the coefficents of α^m, though they do still amount to substitutions of an operator basis for the vector basis. For the 0-weight map the replacement

$$v(x) \to V_{(0)}(x, y) \equiv v(x - y) = \sum_m \alpha^m e^{im(x-y)} \tag{345}$$

gives a zero-weight operator, while a self-dual operator arises by replacing e^{imx} by the self-dual basis operator $\cos mx\ e^{-imy}$,

$$V_{SD}(x, y) = \frac{1}{2}(v(x - y) + v(-(x + y))) = \sum_m \alpha^m \cos mx\ e^{-imy} \tag{346}$$

Each of these $1-1$ relationships between vectors and operators allows us to define an inner product on the operators via the vectors, defining

$$\langle V, W \rangle = \langle \Psi v, \Psi w \rangle \equiv \langle v, w \rangle \tag{347}$$

for Ψ and

$$\left\langle V_{(0)}, W_{(0)} \right\rangle \equiv 2 \left\langle V_{SD}, W_{SD} \right\rangle \equiv \langle v, w \rangle = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \bar{v}^*(x)w(x)dx = \sum (c^{-n})^* d^n \qquad (348)$$

on the 0-weight and self-dual operators by using the vector inner product.

Elementary calculations show how to write this invariant directly in terms of integrals of the various operators. For Ψ, the inner product may be expressed as the full parameter space integral

$$\langle V, W \rangle = \frac{1}{4\pi^2} \int \int \bar{V}^\dagger W \; dxdy \qquad (349)$$

while for the 0-weight and self-dual correspondences, the result is a trace

$$\langle V, W \rangle = tr(\bar{V}^\dagger W) = \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\pi}^{\pi} \bar{V}^*(x, y)W(x, y)dxdy \qquad (350)$$

In these expressions the adjoint dagger is the complex conjugate transpose, $V^\dagger(x, y) = V^{t*}(x, y) = V^*(y, x)$, while the bar denotes the conformal dual. The conformal dual of an operator, $\bar{V}$, is defined so that the dual vector $\overline{Vf}$ is the product of the duals, $\bar{V}\bar{f}$. It follows that for both weight superpositions and self dual operators the dual is found by replacing α^m by α^{-m},

$$\bar{V}_{(0)}(x, y) = \sum \alpha^{-m} e^{im(x-y)} \qquad (351)$$

$$\bar{V}_{SD}(x, y) = \sum \alpha^{-m} \cos mx \; e^{-imy} \qquad (352)$$

The form $\bar{V}^\dagger W$ may be inferred directly by considering the inner product of the vectors Vf and Wg for arbitrary f and g. Thus,

$$\langle Vf, Wg \rangle = \left(\frac{1}{2\pi}\right)^3 \int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \overline{(V(x, y)f(y))}^* (W(x, z)g(z)) \, dxdydz \qquad (353)$$

$$= \left(\frac{1}{2\pi}\right)^3 \int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \bar{V}^*(x, y)\bar{f}^*(y) (W(x, z)g(z)) \, dxdydz \qquad (354)$$

$$= \left\langle f, \bar{V}^\dagger Wg \right\rangle \qquad (355)$$

We can extend the inner product on self-dual operators to other operators as well. For anti-self-dual operators, the conformal dual is also given by $\bar{V}(\alpha^m) = V(\alpha^{-m})$, but now the norm $\langle Vf, Vf \rangle$ is negative for every f. To maintain positivity of the norm for anti-self-dual operators, we therefore assign a multiplicative grading, $\eta = 0$ for self-dual and $\eta = 1$ for anti-self-dual operators, and write

$$\langle V, W \rangle = (-)^{\eta_V \eta_W} tr(\bar{V}^\dagger W) \qquad (356)$$

where

$$\bar{V} = (-)^{\eta_V} V \tag{357}$$

Finally, we can consider the inner product of a self-dual with an anti-self-dual operator. Here the grading proves irrelevant, since the product always vanishes

$$
\begin{aligned}
\langle V_{SD}, W_{ASD} \rangle &= (-)^{0 \cdot 1} tr(\bar{V}_{SD}^{\dagger} W_{ASD}) & (358) \\
&= \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \sum (\alpha^{-m})^* \cos mx \ e^{imy} \sum (\beta^n) \sin nx \ e^{-iny} dx dy & (359) \\
&= 0 & (360)
\end{aligned}
$$

Thus, while the Hilbert norm is defined only for self-dual $(0,1)$ vectors, the norm of $(r,2)$ tensors may be extended to include products of either self-dual or anti-self-dual objects. This extension is useful for $r > 0$ because it allows us to consistently define the Lorentz inner product without requiring an immersion of the parameter space into spacetime. For example, to determine whether the $(1,2)$ tensor

$$V^\mu(x,y) \tag{361}$$

represents a timelike, null or spacelike vector we need to make a scalar of the product

$$V^\mu(x,y)V_\mu(u,v) \tag{362}$$

and it is important to do this in a way that doesn't bias the signs. The grading above accomplishes this because

$$
\begin{aligned}
\langle V_{SD}f, V_{SD}f \rangle &= \left\langle f, \bar{V}_{SD}^{\dagger} V_{SD} f \right\rangle \geq 0 & (363) \\
\langle V_{ASD}f, V_{ASD}f \rangle &= \left\langle f, \bar{V}_{ASD}^{\dagger} V_{ASD} f \right\rangle \leq 0 & (364)
\end{aligned}
$$

for all vectors f. It is important to note that the inner product of operator-valued vectors which are not orthogonal in spacetime may nontheless vanish due to orthogonality of the operators. Thus, some familiar mnemonics such as the nonvanishing of the inner product of any two timelike vectors will no longer hold when such vectors are operator-valued.

A general operator may be decomposed into orthogonal parts

$$R = \frac{1}{2}(\bar{R} + R) + \frac{1}{2}(R - \bar{R}) = R_{SD} + R_{ASD} \tag{365}$$

The inner product of two such operators

$$\langle R, S \rangle = \langle R_{SD}, S_{SD} \rangle - \langle R_{ASD}, S_{ASD} \rangle \tag{366}$$

is then positive definite, $\langle R, R \rangle \geq 0$. There is no inherent conflict between the positivity of this norm on operators and the existence of maps to the indefinite norm vector space,

described at the start of this Appendix. There are other maps possible which preserve the positive norm. For example, we can map a general operator of the form

$$R = R_{SD} + R_{ASD} = \sum (\alpha^m \cos mx + \beta^m \sin mx)\, e^{-imy} \tag{367}$$

to the self-dual vector

$$r = \sum (\alpha^m \cos 2mx + \beta^m \cos(2m+1)x) \tag{368}$$

so that both R and r have positive norm. This map is clearly $1-1$.

Next, we consider the self-dual operator norm on equivalence classes of self-dual operators. The equivalence classes exist because there are many self-dual mappings that act in the same way on all self-dual vectors. Unfortunately, varying over the set of operators equivalent in their effect on self-dual vectors also varies the operator norm. The problem may be resolved by maximizing the norm over the equivalence class.

Consider the action of two self-dual operators

$$U = \sum \alpha^m \cos mx\, e^{-imy} \tag{369}$$
$$V = \sum \beta^m \cos mx\, e^{-imy} \tag{370}$$

on a self-dual vector

$$f = \sum c^k e^{ikx} \tag{371}$$

where $c^k = c_k$. The result is

$$Uf = \sum \alpha^m c^m \cos mx = \alpha^0 c^0 + \frac{1}{2}\sum_{m\neq 0}(\alpha^m + \alpha_m)c^m e^{imx} \tag{372}$$

for U, and a similar result for Vf. Equating Uf and Vf gives

$$\alpha^0 c^0 + \frac{1}{2}\sum_{m\neq 0}(\alpha^m + \alpha_m)c^m e^{imx} = \beta^0 c^0 + \frac{1}{2}\sum_{m\neq 0}(\beta^m + \beta_m)c^m e^{imx} \tag{373}$$

or, since equality must hold for all convergent sequences c^m,

$$\alpha^m + \alpha_m = \beta^m + \beta_m \qquad \forall m \tag{374}$$

We can parameterize the entire class by introducing a parameter $\lambda_m \in [0,1]$. Then, starting with the member of the class having $\alpha_m = 0$ for all $m \neq 0$, write

$$\beta^m = \lambda_m \alpha^m \tag{375}$$
$$\beta_m = (1 - \lambda_m)\alpha^m \tag{376}$$

while for $m = 0$ we always have $\alpha^0 = \beta^0$. The norm of a generic member of the equivalence class is then

$$\langle V, V \rangle = (\alpha^0)^* \alpha^0 + \frac{1}{2}\sum_{m\neq 0}\lambda_m(1-\lambda_m)(\alpha^m)^*\alpha^m \tag{377}$$

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We take the maximum of this norm as defining of the norm for the class. The maximum is found by extremizing with respect to λ_m for each m

$$0 = \frac{\partial \langle V, V \rangle}{\partial \lambda_m} = (1 - 2\lambda_m)(\alpha^m)^* \alpha^m \tag{378}$$

so that

$$\lambda_m = \frac{1}{2} \tag{379}$$

The norm of the class is therefore given by the norm of

$$V_{\max} = \beta^0 + \sum_{m \neq 0} \beta^m \cos mx \ e^{-imy} \tag{380}$$

where $\beta_m = \beta^m$, given by

$$\langle V_{\max}, V_{\max} \rangle = (\beta^0)^* \beta^0 + \frac{1}{2} \sum_{m \neq 0} (\beta^m)^* \beta^m \tag{381}$$

This norm has the convenient property that each term is positive definite. In dealing with self-dual operator norms, we will always choose the representative $V = V_{\max}$.

As a simple example of the operator inner product, we compute the inner products of the $(1, 2)$ tensors P^μ and Q^μ of Sec.(8). Noting that Q^μ is anti-self-dual while P^μ is self-dual, we compute the possible inner products in accordance with the rules developed above:

$$\langle Q^\mu, Q^\nu \rangle = -tr((\bar{Q}^\mu_{\max})^\dagger Q_{\max \ \mu}) \tag{382}$$

$$= -\frac{1}{4\pi^2} \int \int dx dy \left[\bar{Q}^\mu_{\max}(x, y) \right]^* Q_{\max \ \mu}(x, y) \tag{383}$$

$$= -\frac{1}{4\pi^2} \int \int dx dy \overline{\left(\sum_{m \neq 0} \alpha^\mu_m \sin(mx) \ e^{-imy} \right)}^* \left(\sum_{n \neq 0} \alpha_{\mu n} \sin(nx) \ e^{-iny} \right) \tag{384}$$

$$= -\frac{1}{2\pi} \int dx \sum_{m \neq 0} (\alpha^{\mu m})^* \alpha^{-m}_\mu \sin^2(mx) \tag{385}$$

$$= -\frac{1}{2} \sum_{m \neq 0} (\alpha^{\mu m})^* \ \alpha_{\mu m} \tag{386}$$

$$= -\frac{1}{2} \sum_{m \neq 0} (\alpha^{\mu m})^* \ \alpha^m_\mu \tag{387}$$

where the final step follows because the norm is given by that of $Q^\mu = Q^\mu_{\max}$. For the norm of the self-dual operator P^μ we find

$$\langle P^\mu, P^\nu \rangle = tr((\bar{P}^\mu_{\max})^\dagger P_{\max \ \mu}) \tag{388}$$

$$= (\alpha^{\mu 0})^* \ \alpha_{\mu 0} + \frac{1}{2} \sum_{m \neq 0} (\alpha^{\mu m})^* \ \alpha^m_\mu < 0 \tag{389}$$

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where the inequality follows because P^μ is timelike. Finally,

$$\langle P^\mu, Q^\nu \rangle = 0 \tag{390}$$

since P^μ and Q^μ have opposite duality.

This provides an example of the case alluded to above, where both of the $(1,2)$ vectors $u^\mu = \alpha_0^\mu$ and $v^\mu = P^\mu - \alpha_0^\mu$ are timelike, $u^\mu u_\mu = \alpha_0^\mu \alpha_{\mu 0} < 0$ and $v^\mu v_\mu = \sum_{m \neq 0} (\alpha_m^\mu)^* \alpha_{\mu m} < 0$, even though they are orthogonal, $u_\mu v^\mu = 0$. This happens because $P^\mu = u^\mu + v^\mu$ is an orthogonal operator decomposition.

We conclude with two particular types of self-dual maps: identity maps and constant maps. Identity maps are those which map every self-dual vector f to a multiple of f,

$$J_{SD} f = \lambda f \tag{391}$$

They are necessarily of the form

$$J_{SD}(x, y) = \frac{1}{2} \lambda (J_0 + K_0) \tag{392}$$

There are also rank $(r, 2)$ tensor valued identity maps J_{SD}, which take the form

$$J_{SD}^{\mu \cdots \nu}(x, y) = \frac{1}{2} \lambda^{\mu \cdots \nu} (J_0 + K_0) \tag{393}$$

Constant maps are those that map every vector f to the same constant value,

$$C_{SD} f = c \tag{394}$$

These maps consist of the zero-mode only,

$$C_{SD} = \sum \alpha_m \cos mx \; e^{imy} = \alpha_0 \tag{395}$$

and also have genereralizations to $(r, 2)$ tensors given by

$$C_{SD}^{\mu \cdots \nu} = \alpha_0^{\mu \cdots \nu} \tag{396}$$

Notice that the norm of an equivalence class of self-dual maps V_{SD} is

$$\langle V_{SD}, V_{SD} \rangle = (\alpha^0)^* \alpha^0 \tag{397}$$

if and only if V is a constant map. For proof, we have that the norm of V_{SD} is equal to the norm of V_{max}, which is $(\alpha^0)^* \alpha^0 + \frac{1}{2} \sum_{m \neq 0} (\alpha^m)^* \alpha^m$. Since the norm is also $(\alpha^0)^* \alpha^0$ and $(\alpha^m)^* \alpha^m$ is positive definite, we must have $\alpha^m = 0$ for all $m \neq 0$ in V_{max}, so $V_{max} = V_{SD} = \alpha^0$.

Appendix 5: Vector-valued operator algebras   The maximally and minimally commuting complete Lie algebras of definite weight operators were shown in Secs.(3) and (4) to be the mode and Virasoro algebras, respectively. It is a simple matter to extend these algebras to $(1,2)$ tensor algebras of operators. Suppose we have definite weight operators $G_{(m)}$ satisfying

$$[G_{(m)}, G_{(n)}] = a_{mn} G_{(m+n)} + c_{mn}\mathbf{1} \tag{398}$$

Now consider the possible corresponding algebra for definite weight $(1,2)$ tensors $H^a_{(m)}$. Clearly

$$[H^a_{(m)}, H^b_{(n)}] = a_{mn} T^{ab}{}_c\, H^c_{(m+n)} + c_{mn} S^{ab}\mathbf{1} \tag{399}$$

where $T^{ab}{}_c$ and S^{ab} are Lorentz tensors, symmetric on ab. These generalizations are only possible if the Lorentz part of the tangent space under consideration has appropriate tensor fields $T^{ab}{}_c$ and S^{ab}. Unless the spacetime has nonvanishing torsion, there is no rank-3 tensor available, but there is always the Lorentz metric for S^{ab}. Therefore, the mode algebra generalizes immediately to

$$[J^a_{(m)}, J^b_{(n)}] = m\delta^0_{m+n}\eta^{ab}\mathbf{1} \tag{400}$$

while there is no vector-valued generalization of the Virasoro algebra.

The existence of this algebra permits us to add a map from $(1,1)$ tensors to $(1,2)$ tensors, parallel to the maps of $(0,1)$ tensors to $(0,2)$ tensors presented in Appendix 4. Starting with

$$v^a(x) = \sum v^{am} e^{imx} \tag{401}$$

we can map to the superpostion of definite weight operators

$$V^a(x,y) = v^a(x) J_0(x,y) = \sum_{m,n} v^{am} e^{imx} e^{in(x-y)} = \sum_m v^{am} J_m(x,y) \equiv \sum_m V^a_m \tag{402}$$

In the absence of central charges, the V^a commute

$$[V^a, V^b] = \sum_{m,n} v^{am} v^{bn} (e^{imx} e^{iny} - e^{imy} e^{inx}) \sum_k e^{ik(x-y)} = 0 \tag{403}$$

where the sums vanish because $\sum_k e^{ik(x-y)} = 2\pi\delta(x-y)$.

Naturally the definite weight operators V^a_m also commute, up to the central charges allowed by projective representations. To treat the central charges, we need to establish a basis to relate the coefficients V^a_m to the ones in the definition of the J^a_m algebra. Let a basis algebra J^a_m be defined by choosing a vielbein, $e_{m\mu}{}^a$ for each m. Then, for each m, we have commuting operators

$$J^a_{\mu(m)} = \sum_n e_{m\mu}{}^a e^{imx} e^{in(x-y)} \tag{404}$$

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With the most general allowed central charges, the $J^a_{\mu(m)}$ satisfy

$$[J^a_{\mu(m)}, J^b_{\nu(n)}] = m\delta^0_{m+n}(\lambda_1 \eta^{ab} g_{\mu\nu} + \lambda_2 \varepsilon^{ab}{}_{cd}\, e_{m\mu}{}^c\, e_{n\nu}{}^c)\mathbf{1} \tag{405}$$

where for particular physical fields parity considerations may constrain λ_1 or λ_2 to vanish. In any case, we can expand general vectors $V^a_{(m)}$ in terms of the orthonormal basis

$$V^a_{(m)} = V^\mu_{(m)} e_{m\mu}{}^a \tag{406}$$

Then the central charges,

$$
\begin{aligned}
[V^a_{(m)}, V^b_{(n)}] &= V^\mu_{(m)} V^\nu_{(n)}[J^a_{\mu(m)}, J^b_{\nu(n)}] \tag{407}\\
&= m(\lambda_1 V^{\mu m} V_{\mu m}\eta^{ab} + \lambda_2 \varepsilon^{ab}{}_{cd} V^c_{(m)} V^d_{(-m)})\delta^0_{m+n}\mathbf{1} \tag{408}
\end{aligned}
$$

are linear combinations of the combined conformally invariant and Lorentz invariant magnitude $V^{\mu m} V_{\mu m}$ and of the invariant antisymetric product $\varepsilon^{ab}{}_{cd} V^c_{(m)} V^d_{(-m)}$. The second term vanishes for self-dual fields.

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